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SOLUTIONS OF THE EXAMPLES

IN

HALL AND KNIGHT'S

ELEMENTARY TRIGONOMETRY.



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ELEMENTARY TRIGONOMETRY

BY

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ELEMENTARY TRIGONOMETRY.

EXAMPLES. I. PAGE 4.

7. $69^{\circ} 13' 30'' = \cdot 76916$ of a right angle $= 76^{\circ} 91' 66\cdot 7''$.
8. $19^{\circ} 0' 45'' = \cdot 21125$ of a right angle $= 21^{\circ} 12' 50''$.
9. $50^{\circ} 37' 5\cdot 7'' = \cdot 562425$ of a right angle $= 56^{\circ} 24' 25''$.
10. $43^{\circ} 52' 38\cdot 1'' = \cdot 487525$ of a right angle $= 48^{\circ} 75' 25''$.
11. $11^{\circ} 0' 38\cdot 4'' = \cdot 1223407$ of a right angle $= 12^{\circ} 23' 40\cdot 7''$.
12. $142^{\circ} 15' 45'' = 1\cdot 5806944$ of a right angle $= 158^{\circ} 6' 94\cdot 4''$.
13. $12' 9'' = \cdot 00225$ of a right angle $= 22' 50''$.
14. $3' 26\cdot 3'' = \cdot 000636$ of a right angle $= 6' 36\cdot 7''$.
15. $56^{\circ} 87' 50'' = \cdot 56875$ of a right angle $= 51^{\circ} 11' 15''$.
16. $39^{\circ} 6' 25'' = \cdot 390625$ of a right angle $= 35^{\circ} 9' 22\cdot 5''$.
17. $40^{\circ} 1' 25\cdot 4'' = \cdot 4001254$ of a right angle $= 36^{\circ} 0' 40\cdot 6''$.
18. $1^{\circ} 2' 3'' = \cdot 010203$ of a right angle $= 55' 5\cdot 8''$.
19. $3^{\circ} 2' 55'' = \cdot 030205$ of a right angle $= 2^{\circ} 43' 6\cdot 4''$.
20. $8^{\circ} 10' 6\cdot 5'' = \cdot 0810065$ of a right angle $= 7^{\circ} 17' 26\cdot 1''$.
21. $6' 25'' = \cdot 000625$ of a right angle $= 3' 22\cdot 5''$.
22. $37' 5'' = \cdot 003705$ of a right angle $= 20' 0\cdot 4''$.
23. Let the angles expressed in degrees be A and B ;
then $A + B = \frac{9}{10} \times 80^{\circ} = 72^{\circ}$, and $A - B = 18^{\circ}$.
Hence $A = 45^{\circ}$, $B = 27^{\circ}$.
24. If n is the number of degrees in the angle, $n + \frac{10}{9}n = 152$; whence $n = 72$.

25. Here $\frac{x}{60}$ = number of degrees, and $\frac{y}{100}$ = number of grades in the angle.

Therefore $\frac{x}{60} = \frac{9}{10} \cdot \frac{y}{100}$; whence we obtain $50x = 27y$.

26. Here $\frac{s}{60 \times 60}$ = number of degrees, and $\frac{t}{100 \times 100}$ = number of grades in the angle. Therefore $\frac{s}{36} = \frac{9}{10} \times \frac{t}{100}$; that is, $250s = 81t$.

EXAMPLES. II. PAGE 11.

[The following five solutions will sufficiently illustrate this exercise.]

3. From fig. of Art. 17 we have $d^2 = b^2 + c^2 = 400 + 225 = 625$; whence $a = 25$, and $\sin C = \frac{4}{5}$, $\cos B = \frac{4}{5}$, $\cot C = \frac{3}{4}$, $\sec C = \frac{5}{3}$.

6. Let $a = 15$, $b = 9$; then $c^2 = a^2 - b^2 = (a + b)(a - b) = 24 \times 6$; whence $c = 12$, and $\sin C = \frac{4}{5}$, $\cos C = \frac{3}{5}$, $\tan C = \frac{4}{3}$.

8. In the third fig. of Art. 17, let $AC = 41$, $AB = 9$;
then $BC^2 = 41^2 - 9^2 = (41 + 9)(41 - 9)$;
whence $BC = 40$, and $\sin A = \frac{40}{41}$, $\cot A = \frac{9}{40}$.

10. Here $CD = 2DE$. Hence if $ED = a$, $DC = 2a$, $EC = a\sqrt{5}$. The required ratios may now be written down.

11. From the right-angled $\triangle ABC$ we have $BC = 39$; also from the right-angled $\triangle ACD$ we have $DC = 77$. The required ratios may now be written down.

EXAMPLES. III a. PAGE 17.

Examples 1—25 are too easy to require full solution; the following eight solutions will suffice.

$$10. (1 - \cos^2 \theta) \sec^2 \theta = \sin^2 \theta \times \frac{1}{\cos^2 \theta} = \tan^2 \theta.$$

$$12. \operatorname{cosec} a \sqrt{1 - \sin^2 a} = \frac{1}{\sin a} \times \cos a = \cot a.$$

$$15. (1 - \cos^2 \theta) (1 + \tan^2 \theta) = \sin^2 \theta \sec^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta.$$

$$19. (1 - \cos^2 A) (1 + \cot^2 A) = \sin^2 A \operatorname{cosec}^2 A = 1.$$

20. $\sin a \sec a \sqrt{\operatorname{cosec}^2 a - 1} = \frac{\sin a}{\cos a} \times \cot a = 1.$
22. $\sin^2 \theta \cot^2 \theta + \sin^2 \theta = \sin^2 \theta (1 + \cot^2 \theta) = \sin^2 \theta \operatorname{cosec}^2 \theta = 1.$
23. $(1 + \tan^2 \theta) (1 - \sin^2 \theta) = \sec^2 \theta \cos^2 \theta = 1.$
25. $\operatorname{cosec}^2 \theta \tan^2 \theta - 1 = \frac{1}{\sin^2 \theta} \cdot \frac{\sin^2 \theta}{\cos^2 \theta} - 1 = \sec^2 \theta - 1 = \tan^2 \theta.$
26. First side $= \cos^2 A + \sin^2 A = 1.$
27. First side $= \sec^2 A - \tan^2 A = 1.$
28. First side $= \sin A \cdot \sin A + \cos A \cdot \cos A = \sin^2 A + \cos^2 A = 1.$
29. First side $= \sec A \cdot \sec A - \tan A \cdot \tan A = \sec^2 A - \tan^2 A = 1.$
30. $\sin^4 a - \cos^4 a = (\sin^2 a + \cos^2 a) (\sin^2 a - \cos^2 a)$
 $= \sin^2 a - \cos^2 a = \sin^2 a - (1 - \sin^2 a)$
 $= 2 \sin^2 a - 1.$
- Also $\sin^2 a - \cos^2 a = 1 - \cos^2 a - \cos^2 a = 1 - 2 \cos^2 a.$
31. First side $= (\sec^2 a - 1) (\sec^2 a + 1) = \tan^2 a (2 + \tan^2 a).$
32. First side $= (\operatorname{cosec}^2 a - 1) (\operatorname{cosec}^2 a + 1) = \cot^2 a (\cot^2 a + 2).$
33. First side $= \left(\frac{\sin a}{\cos a} \cdot \frac{1}{\sin a} \right)^2 - \left(\frac{\sin a}{\cos a} \right)^2 = \sec^2 a - \tan^2 a.$
34. First side $= \left(\frac{1}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta} \right)^2 - \left(\frac{\cos \theta}{\sin \theta} \right)^2 = \operatorname{cosec}^2 \theta - \cot^2 \theta.$
35. $\operatorname{cosec}^2 \theta - \cot^2 \theta = 1 = \sec^2 \theta - \tan^2 \theta.$ Transpose.

EXAMPLES. III b. PAGE 19.

4. $\operatorname{vers} \theta \sec \theta = (1 - \cos \theta) \sec \theta = \sec \theta - 1.$
5. First side $= \frac{1}{\cos \theta} - \frac{\sin \theta}{\cos \theta} \sin \theta = \frac{1 - \sin^2 \theta}{\cos \theta} = \frac{\cos^2 \theta}{\cos \theta} = \cos \theta.$
6. First side $= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta}$
 $= \frac{1}{\sin \theta \cos \theta} = \operatorname{cosec} \theta \sec \theta.$
7. First side $= \operatorname{cosec} A \tan A \cos A = \frac{\cos A}{\sin A} \tan A = 1.$
8. First side $= \cos^2 \theta + \sin^2 \theta + 2 \cos \theta \sin \theta + \cos^2 \theta + \sin^2 \theta - 2 \cos \theta \sin \theta$
 $= 1 + 1 = 2.$
9. First side $= 1 + 2 \tan \theta + \tan^2 \theta + 1 - 2 \tan \theta + \tan^2 \theta$
 $= 2 + 2 \tan^2 \theta = 2 (1 + \tan^2 \theta) = 2 \sec^2 \theta.$

$$10. \text{ First side} = \cot^2 \theta - 2 \cot \theta + 1 + \cot^2 \theta + 2 \cot \theta + 1 \\ = 2 \cot^2 \theta + 2 = 2 (\cot^2 \theta + 1) = 2 \operatorname{cosec}^2 \theta.$$

$$11. \text{ First side} = \sin^2 A \operatorname{cosec}^2 A + \cos^2 A \sec^2 A = 1 + 1 = 2.$$

$$12. \text{ First side} = \cos^2 A \times 1 + \sin^2 A \times 1 = 1.$$

$$13. \text{ First side} = \cot^2 a (1 + \cot^2 a) = (\operatorname{cosec}^2 a - 1) \operatorname{cosec}^2 a \\ = \operatorname{cosec}^4 a - \operatorname{cosec}^2 a.$$

$$14. \text{ First side} = \frac{\tan^2 a}{\sec^2 a} \cdot \frac{\operatorname{cosec}^2 a}{\cot^2 a} = \frac{\sin^2 a}{\cos^2 a \sec^2 a} \times \frac{\operatorname{cosec}^2 a \sin^2 a}{\cos^2 a} \\ = \frac{\sin^2 a}{\cos^2 a} = \sin^2 a \sec^2 a.$$

$$15. \text{ First side} = \frac{1 + \sin a + 1 - \sin a}{1 - \sin^2 a} = \frac{2}{\cos^2 a} = 2 \sec^2 a.$$

$$16. \text{ First side} = \frac{\tan a (\sec a + 1) + \tan a (\sec a - 1)}{\sec^2 a - 1} = \frac{2 \tan a \sec a}{\tan^2 a} \\ = \frac{2 \sec a}{\tan a} = \frac{2 \sec a \cos a}{\sin a} = \frac{2}{\sin a} = 2 \operatorname{cosec} a.$$

$$17. \text{ First side} = \frac{1}{1 + \sin^2 a} + \frac{1}{1 + \frac{1}{\sin^2 a}} = \frac{1}{1 + \sin^2 a} + \frac{\sin^2 a}{1 + \sin^2 a} \\ = \frac{1 + \sin^2 a}{1 + \sin^2 a} = 1.$$

$$18. \text{ First side} = \left(\frac{1}{\cos \theta} + \frac{1}{\sin \theta} \right) (\sin \theta + \cos \theta) \\ = \frac{(\sin \theta + \cos \theta) (\sin \theta + \cos \theta)}{\cos \theta \sin \theta} \\ = \frac{\sin^2 \theta + \cos^2 \theta + 2 \cos \theta \sin \theta}{\cos \theta \sin \theta} = \frac{1 + 2 \cos \theta \sin \theta}{\cos \theta \sin \theta} \\ = \frac{1}{\cos \theta \sin \theta} + 2 = \sec \theta \csc \theta + 2.$$

$$20. \text{ First side} = (1 + \cot \theta)^2 - \operatorname{cosec}^2 \theta = 1 + 2 \cot \theta + \cot^2 \theta - \operatorname{cosec}^2 \theta \\ = 1 + 2 \cot \theta - 1 = 2 \cot \theta.$$

$$22. \text{ First side} \\ = \sin^2 A + 2 \sin A \operatorname{cosec} A + \operatorname{cosec}^2 A + \cos^2 A + 2 \cos A \sec A + \sec^2 A \\ = (\sin^2 A + \cos^2 A) + 2 + (\cot^2 A + 1) + 2 + (\tan^2 A + 1) \\ = \tan^2 A + \cot^2 A + 7.$$

$$23. \text{ First side} = (2 \sec^2 A - 1) (2 \operatorname{cosec}^2 A - 1) \\ = 4 \sec^2 A \operatorname{cosec}^2 A - 2 \sec^2 A - 2 \operatorname{cosec}^2 A + 1 \\ = 1 + 4 \sec^2 A \operatorname{cosec}^2 A - 2 (\sec^2 A \operatorname{cosec}^2 A) \quad [\text{Art. 31, Ex. 1.}] \\ = 1 + 2 \sec^2 A \operatorname{cosec}^2 A.$$

$$\begin{aligned}
 24. \quad \text{First side} &= 1 + (\sin^2 A + \cos^2 A) - 2 \sin A + 2 \cos A - 2 \sin A \cos A \\
 &= 2 [1 - \sin A + \cos A - \sin A \cos A] \\
 &= 2 (1 - \sin A) (1 + \cos A).
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \text{First side} &= \sin A \left(1 + \frac{\sin A}{\cos A} \right) + \cos A \left(1 + \frac{\cos A}{\sin A} \right) \\
 &= \sin A \frac{(\cos A + \sin A)}{\cos A} + \cos A \frac{(\sin A + \cos A)}{\sin A} \\
 &= \tan A (\sin A + \cos A) + \cot A (\sin A + \cos A) \\
 &= (\sin A + \cos A) \left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} \right) \\
 &= \frac{(\sin A + \cos A) (\sin^2 A + \cos^2 A)}{\sin A \cos A} \\
 &= \frac{\sin A + \cos A}{\sin A \cos A} = \frac{1}{\cos A} + \frac{1}{\sin A} = \sec A + \operatorname{cosec} A.
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \text{First side} &= \cos \theta (2 \tan^2 \theta + 5 \tan \theta + 2) = \frac{2 \sin^2 \theta}{\cos^2 \theta} \cos \theta + 5 \sin \theta + 2 \cos \theta \\
 &= 2 \left(\frac{\sin^2 \theta}{\cos \theta} + \cos \theta \right) + 5 \sin \theta = \frac{2 (\sin^2 \theta + \cos^2 \theta)}{\cos \theta} + 5 \sin \theta \\
 &= 2 \sec \theta + 5 \sin \theta.
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \text{First side} &= \left(\frac{\sin \theta}{\cos \theta} + \frac{1}{\cos \theta} \right)^2 = \left(\frac{1 + \sin \theta}{\cos \theta} \right)^2 \\
 &= \frac{(1 + \sin \theta) (1 + \sin \theta)}{1 - \sin^2 \theta} = \frac{(1 + \sin \theta) (1 + \sin \theta)}{(1 + \sin \theta) (1 - \sin \theta)} = \frac{1 + \sin \theta}{1 - \sin \theta}.
 \end{aligned}$$

$$28. \quad \text{First side} = \frac{\cos \theta (2 \sin \theta - 1)}{2 \sin^2 \theta - \sin \theta} = \frac{\cos \theta (2 \sin \theta - 1)}{\sin \theta (2 \sin \theta - 1)} = \cot \theta.$$

$$\begin{aligned}
 29. \quad \text{First side} &= \frac{\cot^2 \theta (\sec \theta - 1)}{1 + \sin \theta} - \frac{\sec^2 \theta (1 - \sin \theta)}{1 + \sec \theta} \\
 &= \frac{\cot^2 \theta (\sec^2 \theta - 1) - \sec^2 \theta (1 - \sin^2 \theta)}{(1 + \sin \theta) (1 + \sec \theta)} \\
 &= \frac{\cot^2 \theta \tan^2 \theta - \sec^2 \theta \cos^2 \theta}{(1 + \sin \theta) (1 + \sec \theta)} = 0.
 \end{aligned}$$

$$30. \quad \tan^2 \alpha + \sec^2 \beta = (\sec^2 \alpha - 1) + (\tan^2 \beta + 1) = \sec^2 \alpha + \tan^2 \beta.$$

$$31. \quad \frac{\tan \alpha + \cot \beta}{\cot \alpha + \tan \beta} = \frac{\tan \alpha + \frac{1}{\tan \beta}}{\frac{1}{\tan \alpha} + \tan \beta} = \frac{\tan \alpha \tan \beta + 1}{\tan \beta} \cdot \frac{\tan \alpha}{\tan \alpha \tan \beta + 1} = \frac{\tan \alpha}{\tan \beta}.$$

$$\begin{aligned}
 33. \quad \text{First side} &= \cot \alpha \tan \alpha \tan \beta + \tan \beta \cot \beta \cot \alpha \\
 &= \tan \beta + \cot \alpha.
 \end{aligned}$$

34. First side $= \sin^2 \alpha (1 - \sin^2 \beta) - (1 - \sin^2 \alpha) \sin^2 \beta$
 $= \sin^2 \alpha - \sin^2 \alpha \sin^2 \beta - \sin^2 \beta + \sin^2 \alpha \sin^2 \beta$
 $= \sin^2 \alpha - \sin^2 \beta.$
35. First side $= (1 + \tan^2 \alpha) \tan^2 \beta - \tan^2 \alpha (1 + \tan^2 \beta)$
 $= \tan^2 \beta + \tan^2 \alpha \tan^2 \beta - \tan^2 \alpha - \tan^2 \alpha \tan^2 \beta$
 $= \tan^2 \beta - \tan^2 \alpha.$
36. First side $= \sin^2 \alpha \cos^2 \beta + \cos^2 \alpha \sin^2 \beta + \cos^2 \alpha \cos^2 \beta + \sin^2 \alpha \sin^2 \beta,$
the other terms cancelling ;
this expression $= (\sin^2 \alpha + \cos^2 \alpha) \cos^2 \beta + (\cos^2 \alpha + \sin^2 \alpha) \sin^2 \beta$
 $= \cos^2 \beta + \sin^2 \beta = 1.$

EXAMPLES. III. c. PAGE 23.

1. $\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - \sin^2 A}} = \frac{1}{\sqrt{1 - \frac{1}{4}}} = \frac{2}{\sqrt{3}}.$
 $\cot A = \frac{\cos A}{\sin A} = \frac{\sqrt{1 - \sin^2 A}}{\sin A} = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}.$
2. $\sin A = \frac{\tan A}{\sqrt{1 + \tan^2 A}} \quad [\text{Art. 32, Ex. 1}]$
 $= \frac{4}{3} \div \sqrt{1 + \frac{16}{9}} = \frac{4}{3} \times \frac{3}{5} = \frac{4}{5}.$
 $\cos A = \cot A \cdot \sin A = \frac{3}{4} \cdot \frac{4}{5} = \frac{3}{5}.$
5. $\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{1}{49}} = \sqrt{\frac{48}{49}} = \frac{\sqrt{48}}{7}.$
 $\cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{1}{7} \div \frac{\sqrt{48}}{7} = \frac{1}{\sqrt{48}}.$
6. $\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{7^2}{25^2}} = \frac{24}{25}.$ Therefore $\sec A = \frac{25}{24}.$
 $\tan A = \frac{\sin A}{\cos A} = \frac{7}{25} \times \frac{25}{24} = \frac{7}{24}.$
8. $\operatorname{cosec} a = \sqrt{1 + \cot^2 a}. \quad [\text{Art. 27.}]$
 $\cos a = \cot a \sin a = \frac{\cot a}{\operatorname{cosec} a} = \frac{\cot a}{\sqrt{1 + \cot^2 a}}.$

$$10. \operatorname{cosec} A = \frac{1}{\sin A}; \quad \cos A = \sqrt{1 - \sin^2 A}; \quad \sec A = \frac{1}{\sqrt{1 - \sin^2 A}}.$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1 - \sin^2 A}}; \quad \cot A = \frac{\sqrt{1 - \sin^2 A}}{\sin A}.$$

$$11. \text{ Here } \sin A = \cos A, \text{ so that } \tan A = 1.$$

$$\therefore \operatorname{cosec} A = \sqrt{1 + \cot^2 A} = \sqrt{1 + 1} = \sqrt{2}.$$

$$12. \tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}} = \frac{m}{n} \div \sqrt{1 - \frac{m^2}{n^2}}$$

$$= \frac{m}{n} \times \frac{n}{\sqrt{n^2 - m^2}} = \frac{m}{\sqrt{n^2 - m^2}}.$$

$$13. p^2 \cot^2 \theta = q^2 - p^2; \quad \therefore p^2 (\cot^2 \theta + 1) = q^2.$$

$$\therefore p^2 \operatorname{cosec}^2 \theta = q^2, \text{ so that } \sin \theta = \frac{p}{q}.$$

$$14. \text{ In the diagram of Ex. 2, Art. 33, let } PQ = 2m, PR = m^2 + 1; \text{ then}$$

$$RQ^2 = (m^2 + 1)^2 - (2m)^2 = (m^2 - 1)^2.$$

$$\therefore RQ = m^2 - 1.$$

$$\therefore \tan A = \frac{m^2 - 1}{2m}, \quad \sin A = \frac{m^2 - 1}{m^2 + 1}.$$

$$16. \text{ The expression} = \frac{2 \tan \alpha - 3}{4 \tan \alpha - 9}; \text{ but } \tan \alpha = \sqrt{\frac{169}{25} - 1} = \frac{12}{5}.$$

$$\therefore \text{ the expression} = \frac{\frac{2 \cdot 12}{5} - 3}{\frac{4 \cdot 12}{5} - 9} = \frac{9}{3} = 3.$$

$$17. \text{ The expression} = \frac{p \cot \theta - q}{p \cot \theta + q} = \frac{\frac{p^2}{q} - q}{\frac{p^2}{q} + q} = \frac{p^2 - q^2}{p^2 + q^2}.$$

EXAMPLES. IV. a. PAGE 26.

Let E stand for the expression to be evaluated in each case; then

$$6. E = (1)^2 \times \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{3}} (\sqrt{3})^2 = \frac{3}{2}.$$

$$7. E = (\sqrt{3})^2 + 4 \left(\frac{1}{\sqrt{2}} \right)^2 + 3 \left(\frac{2}{\sqrt{3}} \right)^2 = 3 + \frac{4}{2} + 4 = 9.$$

8. $E = \frac{1}{2} \left(\frac{2}{\sqrt{3}} \right)^2 + (\sqrt{2})^2 - 2 \left(\frac{1}{\sqrt{3}} \right)^2 = \frac{1}{2} \cdot \frac{4}{3} + 2 - \frac{2}{3} = 2.$
 9. $E = \left(\frac{1}{\sqrt{3}} \right)^2 + 2 \left(\frac{\sqrt{3}}{2} \right) + 1 - \sqrt{3} + \left(\frac{\sqrt{3}}{2} \right)^2 = \frac{1}{3} + \sqrt{3} + 1 - \sqrt{3} + \frac{3}{4} = \frac{25}{12}.$
 10. $E = (1)^2 + \frac{1}{2} - \left(\frac{\sqrt{3}}{2} \right)^2 - \frac{3}{4} \left(\frac{1}{\sqrt{3}} \right)^2 = 1 + \frac{1}{2} - \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.$
 11. $E = 3 \left(\frac{1}{\sqrt{3}} \right)^2 + \frac{4}{3} \left(\frac{\sqrt{3}}{2} \right)^2 - \frac{1}{2} (\sqrt{2})^2 - \frac{1}{3} \left(\frac{\sqrt{3}}{2} \right)^2 = 1 + 1 - 1 - \frac{1}{4} = \frac{3}{4}.$
 12. $E = \frac{1}{2} - 1^2 + \frac{3}{4} \left(\frac{1}{\sqrt{3}} \right)^2 + \left(\frac{\sqrt{3}}{2} \right)^2 - \frac{1}{2} = \frac{1}{2} - 1 + \frac{1}{4} + \frac{3}{4} - \frac{1}{2} = 0.$
 13. $E = \frac{1}{3} \left(\frac{\sqrt{3}}{2} \right)^2 - \frac{1}{2} \cdot 2 \left(\frac{1}{\sqrt{3}} \right)^2 + \frac{4}{3} \left(\frac{1}{\sqrt{2}} \right)^2 (\sqrt{3})^2 = \frac{1}{4} - \frac{1}{3} + 2 = 1\frac{1}{2}.$
 14. We have
 15. We have
- $$(1)^2 - \left(\frac{1}{2} \right)^2 = x \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \sqrt{3}; \quad x \cdot \frac{1}{2} \cdot \left(\frac{1}{\sqrt{2}} \right)^2 = \frac{(\sqrt{3})^2 \cdot 2 \cdot 1}{(\sqrt{2})^2 \cdot 2};$$
- $$1 - \frac{1}{4} = x \cdot \frac{\sqrt{3}}{2}; \quad \frac{x}{4} = \frac{3}{2};$$
- $$\therefore x = \frac{\sqrt{3}}{2}. \quad \therefore x = 6.$$

EXAMPLES. IV. b. PAGE 28.

For Examples 9—14, see Example 1, page 28.

20. Second side $= 1 + \sin^2 A \sec^2 A = 1 + \tan^2 A = \sec^2 A = \operatorname{cosec}^2 (90^\circ - A).$
21. First side $= \sin A \cot A \tan A \operatorname{cosec} A = 1.$
22. First side $= \operatorname{cosec} A - \cot A \sin A \cot A = \operatorname{cosec} A (1 - \cos^2 A) = \sin A.$
23. First side $= \tan^2 A \operatorname{cosec}^3 A - \sin^2 A \sec^2 A$
 $= \sec^2 A - \tan^2 A = 1.$
24. First side
 $= \cot A + \tan A = \frac{\cos^2 A + \sin^2 A}{\sin A \cos A} = \operatorname{cosec} A \sec A = \operatorname{cosec} A \operatorname{cosec} (90^\circ - A).$
25. First side $= \frac{\cos A}{\operatorname{cosec} A} \cdot \frac{\cot A}{\cos A} = \cos A.$
26. First side $= \frac{\operatorname{cosec}^2 A \tan^2 A}{\tan A} \cdot \frac{\cot A}{\sec^2 A} = \cot^2 A.$
 $= \operatorname{cosec}^2 A - 1 = \sec^2 (90^\circ - A) - 1.$

$$27. \text{ First side} = \frac{\tan A}{\operatorname{cosec}^2 A} \cdot \frac{\sec A \cot^3 A}{\cos^2 A} = \sec A = \sqrt{\tan^2 A + 1}.$$

$$28. \text{ First side} = \frac{\sin^2 A}{1 - \cos A} = \frac{1 - \cos^2 A}{1 - \cos A} = 1 + \cos A = 1 + \sin (90^\circ - A).$$

$$29. \text{ First side} = \frac{\cot^2 A \cos^2 A}{\cot A (1 + \sin A)} = \cot A (1 - \sin A) = \tan (90^\circ - A) - \cos A.$$

$$30. \quad x \cos A \tan A = \sin A ; \\ \therefore x = 1.$$

$$31. \quad \sec^2 A - x \tan A = 1 ; \\ \therefore \tan^2 A = x \tan A ; \\ \therefore x = \tan A.$$

EXAMPLES. IV. d. PAGE 31.

$$9. \quad 1 + \tan^2 \theta = 2 \tan^2 \theta ; \\ \tan^2 \theta = 1 ; \\ \therefore \tan \theta = \pm 1 ; \\ \therefore \theta = 45^\circ.$$

$$10. \quad 1 + \cot^2 \theta = 4 \cot^2 \theta ; \\ 3 \cot^2 \theta = 1 ; \\ \therefore \cot \theta = \pm \frac{1}{\sqrt{3}} ; \\ \therefore \theta = 60^\circ.$$

$$11. \quad 1 + \tan^2 \theta = 3 \tan^2 \theta - 1 ; \\ 2 \tan^2 \theta = 2 ; \\ \therefore \tan \theta = \pm 1 ; \\ \therefore \theta = 45^\circ.$$

$$12. \quad 1 + \tan^2 \theta + \tan^2 \theta = 7 ; \\ 2 \tan^2 \theta = 6 ; \\ \therefore \tan \theta = \pm \sqrt{3} ; \\ \therefore \theta = 60^\circ.$$

$$13. \quad \cot^2 \theta + 1 + \cot^2 \theta = 3 ; \\ \cot^2 \theta = 1 ; \\ \therefore \cot \theta = \pm 1 ; \\ \therefore \theta = 45^\circ.$$

$$14. \quad 2 (\cos^2 \theta - 1 + \cos^2 \theta) = 1 ; \\ 4 \cos^2 \theta = 3 ; \\ \therefore \cos \theta = \pm \frac{\sqrt{3}}{2} ; \\ \therefore \theta = 30^\circ.$$

$$15. \quad 2 \cos^2 \theta + 4 - 4 \cos^2 \theta = 3 ; \\ 2 \cos^2 \theta = 1 ; \\ \therefore \cos \theta = \pm \frac{1}{\sqrt{2}} ; \\ \therefore \theta = 45^\circ.$$

$$16. \quad 6 \cos^2 \theta - \cos \theta - 1 = 0 ; \\ (3 \cos \theta + 1) (2 \cos \theta - 1) = 0 ; \\ \therefore \cos \theta = \frac{1}{2} \text{ or } -\frac{1}{3} ; \\ \therefore \theta = 60^\circ.$$

$$17. \quad 12 \sin^2 \theta - 4 \sin \theta - 1 = 0 ; \\ (6 \sin \theta + 1) (2 \sin \theta - 1) = 0 ; \\ \therefore \sin \theta = \frac{1}{2} \text{ or } -\frac{1}{6} ; \\ \therefore \theta = 30^\circ.$$

$$18. \quad 2 - 2 \cos^2 \theta = 3 \cos \theta ; \\ 2 \cos^2 \theta + 3 \cos \theta - 2 = 0 ; \\ \therefore (2 \cos \theta - 1) (\cos \theta + 2) = 0 ; \\ \therefore \cos \theta = \frac{1}{2}, \text{ so that } \theta = 60^\circ.$$

$$19. \quad \tan \theta = 4 - \frac{3}{\tan \theta};$$

$$\tan^2 \theta - 4 \tan \theta + 3 = 0;$$

$$\therefore (\tan \theta - 1)(\tan \theta - 3) = 0;$$

$$\therefore \tan \theta = 1 \text{ or } 3;$$

$$\therefore \theta = 45^\circ, \text{ or } 71^\circ 34'.$$

$$21. \quad \frac{1 + \tan^2 \theta}{\tan \theta} = 2 \sec \theta;$$

$$\frac{\sec^2 \theta}{\tan \theta} = 2 \sec \theta;$$

$$\therefore \sec \theta = 0, \text{ or } \frac{\sec \theta}{\tan \theta} = 2;$$

$$\therefore \sin \theta = \frac{1}{2}, \text{ so that } \theta = 30^\circ.$$

$$23. \quad \frac{\sin^2 \theta - \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1}{\sin \theta};$$

$$\sin^2 \theta - \cos^2 \theta = \cos \theta;$$

$$1 - 2 \cos^2 \theta = \cos \theta;$$

$$2 \cos^2 \theta + \cos \theta - 1 = 0;$$

$$\therefore (2 \cos \theta - 1)(\cos \theta + 1) = 0;$$

$$\therefore \cos \theta = \frac{1}{2}, \text{ or } -1;$$

$$\therefore \theta = 60^\circ.$$

$$25. \quad \tan \theta (2 \sin \theta - 1) = 2 \sin \theta - 1;$$

$$(2 \sin \theta - 1)(\tan \theta - 1) = 0;$$

$$\therefore \sin \theta = \frac{1}{2}, \text{ or } \tan \theta = 1;$$

$$\therefore \theta = 30^\circ \text{ or } 45^\circ.$$

$$27. \quad 5 \tan \theta + \frac{6}{\tan \theta} = 11;$$

$$5 \tan^2 \theta - 11 \tan \theta + 6 = 0;$$

$$(5 \tan \theta - 6)(\tan \theta - 1) = 0;$$

$$\therefore \tan \theta = 1, \text{ or } 1.2;$$

$$\therefore \theta = 45^\circ, \text{ or } 50^\circ 12'.$$

$$20. \quad \cos^2 \theta - 1 + \cos^2 \theta = 2 - 5 \cos \theta;$$

$$2 \cos^2 \theta + 5 \cos \theta - 3 = 0;$$

$$\therefore (2 \cos \theta - 1)(\cos \theta + 3) = 0;$$

$$\therefore \cos \theta = \frac{1}{2}, \text{ so that } \theta = 60^\circ.$$

$$22. \quad \frac{4}{\sin \theta} + 2 \sin \theta = 0;$$

$$2 \sin^2 \theta - 9 \sin \theta + 4 = 0;$$

$$\therefore (2 \sin \theta - 1)(\sin \theta - 4) = 0;$$

$$\therefore \sin \theta = \frac{1}{2}, \text{ so that } \theta = 30^\circ.$$

$$24. \quad 2 \cos \theta + 2 \sqrt{2} = \frac{3}{\cos \theta};$$

$$2 \cos^2 \theta + 2 \sqrt{2} \cos \theta - 3 = 0;$$

$$\therefore (\sqrt{2} \cos \theta - 1)(\sqrt{2} \cos \theta + 3) = 0;$$

$$\therefore \cos \theta = \frac{1}{\sqrt{2}};$$

$$\therefore \theta = 45^\circ.$$

$$26. \quad 6 \frac{\sin \theta}{\cos \theta} - \frac{5 \sqrt{3}}{\cos \theta} + 12 \frac{\cos \theta}{\sin \theta} = 0;$$

$$6 \sin^2 \theta - 5 \sqrt{3} \sin \theta + 12 (1 - \sin^2 \theta) = 0;$$

$$6 \sin^2 \theta + 5 \sqrt{3} \sin \theta - 12 = 0;$$

$$\therefore (2 \sin \theta - \sqrt{3})(3 \sin \theta + 4 \sqrt{3}) = 0;$$

$$\therefore \sin \theta = \frac{\sqrt{3}}{2}, \text{ so that } \theta = 60^\circ.$$

$$28. \quad 1 + \tan^2 \theta + \tan^2 \theta = 3 \tan \theta;$$

$$2 \tan^2 \theta - 3 \tan \theta + 1 = 0;$$

$$\therefore (2 \tan \theta - 1)(\tan \theta - 1) = 0;$$

$$\therefore \tan \theta = 1, \text{ or } \frac{1}{2}.$$

$$\therefore \theta = 45^\circ, \text{ or } 26^\circ 34'.$$

MISCELLANEOUS EXAMPLES. A. PAGE 32.

3. If θ be the angle, we have $\sin \theta = \frac{21}{29}$, so that $\operatorname{cosec} \theta = \frac{29}{21}$.

$$\text{Also } \cos \theta = \sqrt{1 - \left(\frac{21}{29}\right)^2} = \frac{\sqrt{(29+21)(29-21)}}{29} = \frac{20}{29}.$$

$$4. \tan A = \frac{1}{\sqrt{\operatorname{cosec}^2 A - 1}} = \frac{15}{\sqrt{2 \times 32}} = \frac{15}{8};$$

$$\sec A = \sqrt{1 + \tan^2 A} = \frac{\sqrt{8^2 + 15^2}}{8} = \frac{17}{8}.$$

$$5. \text{First side} = \operatorname{cosec}^2 A - \cot^2 A - 1 = 0.$$

$$7. b = \sqrt{a^2 + c^2} = \sqrt{1681} = 41.$$

$$\cot A = \frac{c}{a} = \frac{9}{40}; \quad \sec A = \frac{b}{c} = \frac{41}{9}; \quad \sec C = \frac{b}{a} = \frac{41}{40}.$$

$$8. \text{See Article 16.}$$

$$9. \text{First side} = \cos \theta (1 - \cos \theta) \frac{1 + \cos \theta}{\cos \theta} = 1 - \cos^2 \theta = \sin^2 \theta.$$

$$10. \text{We have } \sec^2 \alpha = 1 + \tan^2 \alpha = \frac{1 + \cot^2 \alpha}{\cot^2 \alpha}; \quad \therefore \sec \alpha = \frac{\sqrt{1 + \cot^2 \alpha}}{\cot \alpha}.$$

$$\text{Also } \operatorname{cosec}^2 \alpha = 1 + \cot^2 \alpha; \text{ so that } \operatorname{cosec} \alpha = \sqrt{1 + \cot^2 \alpha}.$$

$$11. \text{First side} = 3 \left(\frac{1}{\sqrt{3}} \right)^2 + \frac{1}{4} \cdot 2 + 5 \cdot 1 - \frac{2}{3} \cdot \left(\frac{\sqrt{3}}{2} \right)^2 = 1 + \frac{1}{2} + 5 - \frac{1}{2} = 6.$$

$$12. \sin \alpha = \frac{1}{\sqrt{1 + \cot^2 \alpha}} = \frac{m}{\sqrt{m^2 + n^2}}; \quad \sec \alpha = \sqrt{1 + \tan^2 \alpha} = \frac{\sqrt{m^2 + n^2}}{n}.$$

$$13. m \text{ sexagesimal minutes} = \frac{m}{60 \times 90} \text{ right angles,}$$

$$n \text{ centesimal minutes} = \frac{n}{100 \times 100} \text{ right angles.}$$

$$\therefore \frac{m}{60 \times 90} = \frac{n}{100 \times 100}; \text{ whence } m = .54n.$$

$$14. \cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}, \text{ since } A \text{ is acute;}$$

$$\therefore \tan A + \sec A = \frac{4}{3} + \frac{5}{3} = 3.$$

$$15. \text{First side} = \tan A \cot A \sin A \cot A = \cos A.$$

$$16. RQ = \sqrt{20^2 + 21^2} = \sqrt{841} = 29;$$

$$\therefore \tan Q = \frac{RP}{PQ} = \frac{20}{21}; \quad \operatorname{cosec} Q = \frac{QR}{RP} = \frac{29}{20}.$$

$$17. \text{ First side} = \frac{\sin^2 \alpha - \cos^2 \alpha}{\sin \alpha \cos \alpha} \cdot \sin \alpha \cos \alpha = 1 - \cos^2 \alpha - \cos^2 \alpha = 1 - 2 \cos^2 \alpha.$$

$$18. \text{ See Art. 39, Ex. 1.}$$

$$19. \text{ Second side} = (\sqrt{3})^2 - 2 \cdot \left(\frac{1}{2}\right)^2 - \frac{3}{4} (\sqrt{2})^2 = 3 - \frac{1}{2} - \frac{3}{2} = 3 - 2 \\ = \tan^2 60^\circ - 2 \tan^2 45^\circ.$$

$$20. \begin{array}{ll} (1) \quad 3 \sin \theta = 2 \cos^2 \theta; & (2) \quad 5 \tan \theta - \sec^2 \theta = 3; \\ \quad 2 \sin^2 \theta + 3 \sin \theta - 2 = 0; & \quad 5 \tan \theta - 1 - \tan^2 \theta = 3; \\ \therefore (2 \sin \theta - 1)(\sin \theta + 2) = 0; & \quad \tan^2 \theta - 5 \tan \theta + 4 = 0; \\ \therefore \sin \theta = \frac{1}{2}, \text{ so that } \theta = 30^\circ. & \therefore (\tan \theta - 1)(\tan \theta - 4) = 0; \\ & \text{whence } \theta = 45^\circ, \text{ or } 75^\circ 58'. \end{array}$$

$$21. \text{ First side} = 1 - (\sec^2 A - \tan^2 A)^2 = 1 - 1 = 0.$$

$$22. \quad 6 \sin^2 \theta - 11 \sin \theta + 4 = 0; \quad \therefore (2 \sin \theta - 1)(3 \sin \theta - 4) = 0; \\ \therefore 2 \sin \theta - 1 = 0; \text{ whence } \theta = 30^\circ; \\ \text{or } 3 \sin \theta - 4 = 0; \text{ whence } \sin \theta = \frac{4}{3}, \text{ which is impossible.}$$

$$23. \quad \tan A + \tan B = \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} = \frac{c^2}{ab}.$$

$$24. \quad c + c^{-1} = \cot A + \tan A = \frac{\cos^2 A + \sin^2 A}{\sin A \cos A} = \sec A \operatorname{cosec} A.$$

$$25. \quad (\sin \theta + 2)(3 \sin \theta - 1) = 0, \text{ whence } \sin \theta = \frac{1}{3} = .3333; \\ \therefore \theta = 19^\circ 28'.$$

EXAMPLES. V. a. PAGE 37.

$$1. \quad c = \sqrt{a^2 - b^2} = \sqrt{16 - 12} = 2.$$

$$\sin C = \frac{c}{a} = \frac{1}{2}; \quad \therefore C = 30^\circ. \quad \sin B = \frac{b}{a} = \frac{\sqrt{3}}{2}; \quad \therefore B = 60^\circ.$$

$$2. \quad a = \sqrt{b^2 - c^2} = \sqrt{144 - 36} = \sqrt{108} = 6\sqrt{3}.$$

$$\sin C = \frac{c}{b} = \frac{1}{2}; \quad \therefore C = 30^\circ; \text{ and } A = 90^\circ - C = 60^\circ.$$

$$3. \quad c = \sqrt{a^2 + b^2} = \sqrt{144 + 48} = \sqrt{192} = 8\sqrt{3}.$$

$$\sin B = \frac{b}{c} = \frac{12}{8\sqrt{3}} = \frac{\sqrt{3}}{2}; \quad \therefore B = 60^\circ; \text{ and } A = 90^\circ - B = 30^\circ.$$

$$4. \quad c = \sqrt{a^2 - b^2} = \sqrt{90 \times 30} = 30\sqrt{3}.$$

$$\sin C = \frac{c}{a} = \frac{\sqrt{3}}{2}; \quad \therefore C = 60^\circ, \quad B = 30^\circ.$$

6. $c = \sqrt{a^2 + b^2} = \sqrt{75 + 3 \times 75} = 10\sqrt{3}$.
 $\sin B = \frac{b}{c} = \frac{\sqrt{3}}{2}$; $\therefore B = 60^\circ$, $A = 30^\circ$.
7. $b = c = 2$; $\therefore B = C = 45^\circ$.
 $a = \sqrt{b^2 + c^2} = 2\sqrt{2}$.
8. $a = \sqrt{b^2 - c^2} = \sqrt{3 \times 36 - 27} = 9$.
 $\sin C = \frac{c}{b} = \frac{1}{2}$; $\therefore C = 30^\circ$, $B = 60^\circ$.
9. $B = 90^\circ - A = 60^\circ$.
 $\frac{b}{a} = \tan B$; $\therefore b = 9\sqrt{3} \cdot \sqrt{3} = 27$.
 $\frac{c}{a} = \sec B$; $\therefore c = 9\sqrt{3} \cdot 2 = 18\sqrt{3}$.
10. $C = 90^\circ - 25^\circ = 65^\circ$.
 $b = a \cos C = 4 \times .4226$;
 $c = a \sin C = 4 \times .9063$.
11. $B = 90^\circ - A = 36^\circ$.
 $a = c \cos B = 8 \times .8090$;
 $b = c \sin B = 8 \times .5878$.
12. $B = 180^\circ - C - A = 90^\circ$.
 $a = b \cos C = 6 \times .4540$;
 $c = b \sin 63^\circ = 6 \times .8910$.
13. $A = 90^\circ - C = 53^\circ$.
 $a = b \cos C = 100 \times .7986$;
 $c = b \sin C = 100 \times .6018$.
14. $C = 180^\circ - A - B = 90^\circ$.
 $a = b \tan A = 20$; $c = b \sec A = 40$.
15. $A = 180^\circ - B - C = 90^\circ$.
 $b = c = 4$; $a = \sqrt{b^2 + c^2} = 4\sqrt{2}$.
16. $A = 180^\circ - B - C = 90^\circ$.
 $b = a \cos C = 4$; $c = a \sin C = 4\sqrt{3}$.
17. $a = b \tan A = \frac{49}{.07} = 700$.
18. $a = c \sin A = 50 \times .62 = 31$.
19. $c = a \tan C = 100 \times .8647 = 86.47$.
20. $a = b \sec C = 200 \times 4.89 = 978$.
21. $C = 90^\circ - A = 54^\circ$.
 $a = c \tan A = 100 \times .73 = 73$;
 $b = c \sec A = 100 \times 1.24 = 124$.
22. $\sin C = \frac{c}{a} = .37$; $\therefore C = 21^\circ 43'$.
 $B = 90^\circ - C = 68^\circ 17'$;
 $b = a \cos C = 100 \times .93 = 93$.
23. $C = 90^\circ - B = 50^\circ 36'$.
 $c = b \cot B = 25 \times 1.2174 = 30.435$;
 $a = b \operatorname{cosec} B = 25 \times 1.5755 = 34.3875$.
 $\therefore B = 50^\circ 24'$.
 $A = 90^\circ - B = 39^\circ 36'$.
24. $c = \sqrt{a^2 + b^2} = \sqrt{124609} = 353$.
25. $\cos A = \frac{22.75}{25} = .91$; whence $A = 24^\circ 30'$. Hence $B = 65^\circ 30'$.
 $a = c \sin A = 25 \times .4147 = 10.37$, approx.

EXAMPLES. V. b. PAGE 39.

$$1. \quad F = 180^\circ - 30^\circ - 120^\circ = 30^\circ = A; \quad 2. \quad c = BD \operatorname{cosec} 30^\circ = 20;$$

$$\therefore CB = CA = 20; \quad a = BD \operatorname{cosec} 45^\circ = 10\sqrt{2}.$$

$$\therefore BD = BC \sin 60^\circ = 10\sqrt{3}.$$

$$3. \quad \text{Since } B + C = 90^\circ; \therefore A = 90^\circ. \quad AB = BD \sec 30^\circ = 10\sqrt{3} \text{ ft.}$$

$$AC = AB \tan 30^\circ = 10 \text{ ft.} \quad AD = AB \sin 30^\circ = 5\sqrt{3} \text{ ft.}$$

$$4. \quad \text{Let } QS \text{ be the perpendicular from } Q \text{ on } PR.$$

$$\text{Then } PR = 8 \sec 60^\circ = 16. \quad SR = 8 \cos 60^\circ = 4. \quad \therefore SP = 16 - 4 = 12$$

$$5. \quad SQ = 36 \tan 53^\circ = 47.77. \quad RQ = 36 \tan 35^\circ = 25.21.$$

$$\therefore RS = SQ - RQ = 22.56.$$

$$6. \quad \text{We have } \angle PRQ = 180^\circ - 135^\circ = 45^\circ; \therefore \angle QPR = 45^\circ;$$

$$\therefore QR = QP = 20, \text{ and } \angle QPS = 90^\circ - 25^\circ = 65^\circ,$$

$$\therefore SQ = 20 \tan 65^\circ = 42.89, \quad \therefore RS = 42.89 - 20 = 22.89.$$

$$7. \quad \text{Let } AD \text{ be the perpendicular and let } AD = x.$$

$$\text{Then } \angle BAD = 90^\circ - 45^\circ = 45^\circ = \angle ABD, \therefore DB = DA = x.$$

$$\text{Now } \frac{DA}{DC} = \tan 60^\circ; \therefore \frac{x}{x - 40} = \sqrt{3}; \therefore x(\sqrt{3} - 1) = 40\sqrt{3}.$$

$$\therefore x = \frac{40\sqrt{3}}{\sqrt{3} - 1} = 20\sqrt{3}(\sqrt{3} + 1) = 20(3 + \sqrt{3}).$$

$$\therefore \text{perpendicular} = 20(3 + \sqrt{3}) = 94.64.$$

$$8. \quad \text{Let } DC = x.$$

$$\text{Since } \angle DCB = 90^\circ - 45^\circ = 45^\circ = \angle CBD; \therefore DB = DC = x.$$

$$\text{And } \frac{DA}{DC} = \cot 35^\circ 18'; \therefore \frac{x + 41.24}{x} = 1.4124.$$

$$\therefore x + 41.24 = 1.4124x; \therefore x = 100;$$

$$\text{that is,} \quad DC = DB = 100.$$

$$9. \quad \text{The perp. } AD = 20 \sin 42^\circ = 20 \times .6691 = 13.382,$$

$$\tan C = \frac{AD}{CD} = \frac{13.382}{18.138} = .7378;$$

$$\text{whence} \quad C = 36^\circ 25'.$$

EXAMPLES. VI. a. PAGE 42.

For Examples 1—5 see figure on page 40.

$$1. \quad \text{Let } BC = \text{height of chimney, } AC = 300 \text{ ft.,}$$

$$\text{then elevation} = \angle BAC = 30^\circ, \therefore BC = AC \tan 30^\circ = 100\sqrt{3} = 173.2 \text{ ft.}$$

2. Let B be the top of the mast, $BC=160$ feet, A the boat observed.
Then $\angle BAC=30^\circ$, $\angle ABC=60^\circ$.

$$\therefore \text{distance required} = AC = BC \tan 60^\circ = 160\sqrt{3} = 277.12 \text{ ft.}$$

3. Let BC represent the pole, and AC its shadow.

$$\text{Then } \tan A = \frac{BC}{AC} = \frac{6}{2\sqrt{3}} = \sqrt{3}; \therefore \text{angle of elevation} = 60^\circ.$$

4. Let BC represent the tower; A the position of the observer.

$$\text{Then } AC = 86.6 \text{ ft. and } \angle BAC = 30^\circ.$$

$$\therefore \text{height of tower} = AC \tan 30^\circ = \frac{86.6}{\sqrt{3}} = 50 \text{ ft.}$$

$$\text{Distance } AB = BC \operatorname{cosec} 30^\circ = 2BC = 100 \text{ ft.}$$

5. Let AB represent the ladder, and BC the wall.

$$\text{Then } AB = 45 \text{ ft. } \angle ABC = 60^\circ.$$

$$\therefore \text{height of wall} = BC = AB \cos 60^\circ = 22.5 \text{ ft.}$$

$$\text{Distance } AC = AB \sin 60^\circ = \frac{45\sqrt{3}}{2} = 38.97 \text{ ft.}$$

6. See figure on page 9.

Let DE , BC represent the masts, and OCE the horizon.

$$\text{Then } \angle BOC = 33^\circ 41'. \quad BC = 40 \text{ ft. } DE = 60 \text{ ft.}$$

$$\text{And } OC = BC \cot 33^\circ 41' = 60 \text{ ft. } OE = DE \cot 33^\circ 41' = 90 \text{ ft.}$$

$$\therefore \text{distance required} = OE - OC = 30 \text{ ft.}$$

7. See figure on page 40.

Let BC represent the cliff and A the observer.

$$\text{Then } \angle BAC = 41^\circ 18'. \quad BC = 132 \text{ yds.}$$

$$\therefore \text{distance required} = AB = BC \operatorname{cosec} 41^\circ 18' = \frac{132}{.66} = 200 \text{ yds.}$$

8. See figure on page 9.

Let BC , DE represent the chimneys, and O the observer.

$$\text{Then } OC = 100 \text{ yds. } \angle BOC = 27^\circ 2'.$$

$$\text{Now } BC = OC \tan 27^\circ 2' = 51 \text{ yds.}$$

$$\therefore DE = BC + 30 \text{ yds.} = 81 \text{ yds.}$$

9. See figure on page 41.

Let PT be the tower, and Q , R the two points of observation.

$$\text{Then } \angle PQR = 30^\circ, \quad \angle PRT = 60^\circ, \quad QR = 100 \text{ yds.}$$

$$\therefore \angle RPQ = 60^\circ - 30^\circ = 30^\circ = \angle PQR, \quad \therefore RP = RQ = 100 \text{ yds.}$$

$$\therefore \text{height of tower} = RP \sin 60^\circ = 50\sqrt{3} = 86.6 \text{ yds.}$$

10. Let AB be the flagstaff, BC the building, D the point of observation.

Then $DC = 40$ ft. $\angle ADC = 60^\circ$, $\angle BDC = 30^\circ$;

$$\therefore \angle DAB = 30^\circ = \angle ADB; \quad \therefore BA = BD = 40 \sec 30^\circ = \frac{80}{\sqrt{3}} = 49.19 \text{ ft.}$$

11. See figure on page 41.

Let PT be the spire, R, Q the two points of observation.

Then $QR = 200$ ft. $\angle PRT = 45^\circ$, $\angle PQR = 30^\circ$.

$$\angle RPT = 45^\circ; \quad \therefore TP = TR.$$

Let x ft. = height of spire.

$$\text{Then } \frac{x}{x+200} = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

$$\therefore x = \frac{200}{\sqrt{3}-1} = 100(1+\sqrt{3}) = 273.2 \text{ feet.}$$

12. See figure on page 43.

Let CD represent the post, and AB the steeple.

Then $CD = 30$ ft. $\angle ACE = 30^\circ$, $\angle ADB = 45^\circ$;

$$\therefore \angle DAB = 45^\circ = \angle ADB, \quad \therefore BA = BD = x \text{ feet, say.}$$

$$\therefore \tan 30^\circ = \frac{AE}{EC} = \frac{x-30}{x}; \quad \therefore x(\sqrt{3}-1) = 30\sqrt{3}. \quad \therefore x = 70.98 \text{ ft.}$$

That is height = distance = 70.98 ft.

13. Let B be the top of the hill, and C the point on the horizontal plane vertically below B . Let D be the position of the balloon when the observation is made. Draw DE perpendicular to BC .

Then $BC = 3300$ ft. $\angle BDE = 30^\circ$, $\angle BAC = 60^\circ$;

$$\therefore AC = BC \cot 60^\circ = 1100\sqrt{3} \text{ feet.}$$

And $BE = DE \tan 30^\circ = AC \tan 30^\circ = 1100$ feet.

$$\therefore DA = EC = 3300 - 1100 = 2200 \text{ feet.}$$

$\therefore$ the balloon rises 2200 feet in 5 minutes,

that is, $\frac{2200 \times 60}{1760 \times 3 \times 5}$ miles per hour, or 5 miles per hour.

14. See figure on page 44.

Let OA represent the monument, B, C the two objects, OP the horizontal line through O ;

Then $\angle POC = 30^\circ$; $\therefore \angle OCB = 30^\circ$.

$$\angle POB = 45^\circ; \quad \therefore \angle BOA = \angle OBA = 45^\circ; \quad \therefore AO = AB = 100 \text{ feet.}$$

Let x feet = CB = required distance.

$$\text{Then } \frac{100}{x+100} = \tan 30^\circ = \frac{1}{\sqrt{3}}; \quad \therefore x = 100(\sqrt{3}-1);$$

$\therefore$ distance required = 73.2 feet.

15. See figure on page 43.

Let AB represent the monument, CD the tower.

Then $AB = 96$ feet, and the angles are as in the figure;

$$\therefore DB = AB \cot 60^\circ = 32\sqrt{3} \text{ feet};$$

$$\therefore AE = CE \tan 30^\circ = DB \tan 30^\circ = 32 \text{ feet.}$$

$$\therefore \text{height of tower} = CD = EB = 96 - 32 = 64 \text{ feet.}$$

16. See figure on page 44.

Let OA represent the cliff, B, C the two boats.

Then $OA = 150$ ft., $\angle OBA = 30^\circ$, $\angle OCB = 15^\circ$;

$$\therefore \angle BOC = 15^\circ, \text{ and } BC = BO.$$

$$\therefore \text{required distance} = BC = BO = AO \operatorname{cosec} 30^\circ = 300 \text{ ft.}$$

17. See figure on page 44.

Let O represent the top of the hill, B, C the milestones.

Then $\angle OBA = 45^\circ$, $\angle OCB = 22^\circ$; $\therefore \angle BOA = 45^\circ$, so that $AO = AB$.

Let $AO = AB = x$ yards.

$$\text{Then } \frac{AC}{AO} = \frac{x + 1760}{x} = \cot 22^\circ = 2.475; \therefore 1.475x = 1760.$$

$$\therefore \text{height of hill} = x = 1193 \text{ yds. nearly.}$$

18. See figure on page 44.

Let OA represent the lighthouse, B, C the two rocks.

Then $OA = 80$ yds., $\angle OBA = 75^\circ$, $\angle OCB = 15^\circ$, $\angle COA = 75^\circ$;

$$\therefore AB = OA \cot 75^\circ = 80 \times .268 \text{ yds.}$$

Let $CB = x$ yds.

Then $x + 80 \times .268 = OA \cot 15^\circ = 80 \times 3.732$, $\therefore x = 80 \times 3.464 = 277.12$ yds.

$$\therefore \text{required distance} = 277.12 \text{ yds.}$$

EXAMPLES. VI b. PAGE 47.

1. Let A, B be the two positions of the observer, P, Q the two objects. Then $AB = 800$ yds., and PQA is a straight line making $\angle PAB$ equal to 45° . Also $\angle PBA = 90^\circ$, $\angle QBA = 45^\circ$. $\therefore QA = QB = QP$.

$$\text{And } QA = AB \cos 45^\circ = \frac{800}{\sqrt{2}} = 565.6 \text{ yds. } PA = 2QA = 1131.2 \text{ yds.}$$

Thus the required distances are 565.6 yds., 1131.2 yds.

2. Let A, B be the two positions of the observer, P, Q the two ships; then APQ is a straight line at right angles to AB .

And $AB = 3$ miles, $\angle ABP = 30^\circ$, $\angle ABQ = 60^\circ$.

$$\therefore BP = AB \sec 30^\circ = \frac{6}{\sqrt{3}} = 3.464 \text{ miles, } BQ = AB \sec 60^\circ = 6 \text{ miles.}$$

Thus the required distances are 3.464 miles, 6 miles.

3. Let O represent the harbour and ON, OE, OS, OW the directions of North, South, East, West.

Let P, Q be the positions of the two ships at 2 p.m.

Then $\angle POW = 28^\circ, \angle QOE = 62^\circ; \therefore \angle POQ = 90^\circ$.

Also $OP = 2 \times 10 = 20$ miles. $OQ = 2 \times 10\frac{1}{2} = 21$ miles.

$$\therefore \text{distance} = PQ = \sqrt{20^2 + 21^2} = 29 \text{ miles.}$$

4. Let O be the position of the lighthouse, and P, Q the points at which the steamer enters and leaves the light.

Then PQ lies East and West, and OP, OQ are the directions of N.E., N.W.

$$\therefore \angle POQ = 90^\circ, \angle PQO = \angle QPO = 45^\circ, OP = OQ = 5 \text{ miles.}$$

$$\therefore PQ = \sqrt{25 + 25} = 5\sqrt{2} \text{ miles. } \therefore \text{steamer sails } 5\sqrt{2} \text{ miles in } 30\sqrt{2} \text{ minutes,}$$

that is, the speed of steamer is 10 miles per hour.

5. Let O, P, Q be the first positions of the ship and lighthouses. OA the direction in which the ship is sailing, A its second position.

$$\text{Then } OA = 10 \text{ miles, } \angle OAP = 45^\circ, \angle AOP = 90^\circ, \angle PAQ = 22\frac{1}{2}^\circ.$$

$$\therefore \angle PQA = 45^\circ - 22\frac{1}{2}^\circ = 22\frac{1}{2}^\circ = PAQ; \therefore PA = PQ.$$

$$\text{And } OP = OA = 10 \text{ miles, } PA = OA \sec 45^\circ = 10\sqrt{2} \text{ miles.}$$

$$\therefore OQ = OP + PQ = OP + PA = 10(\sqrt{2} + 1) = 24.14 \text{ miles.}$$

$$\therefore \text{distances are } 10 \text{ miles, } 24.14 \text{ miles.}$$

6. As before let O be the port, and ON, OE, OS, OW the directions of the cardinal points of the compass.

Let P, Q be the positions of the ships at the end of an hour.

$$\text{Then } OP = 8 \text{ miles, } OQ = 8\sqrt{3} \text{ miles.}$$

$$\text{And } \angle PON = 35^\circ, \angle QOS = 55^\circ; \therefore \angle POQ = 90^\circ.$$

$$\therefore \text{distances apart} = PQ = \sqrt{64 + 3 \times 64} = 16 \text{ miles.}$$

$$\text{Also } \tan QPO = \frac{8\sqrt{3}}{8} = \sqrt{3}; \therefore \angle QPO = 60^\circ.$$

$$\therefore \angle QPO - \angle PON = 60^\circ - 35^\circ = 25^\circ;$$

$$\therefore \text{bearing of the second vessel as observed from the first is S. } 25^\circ \text{ W.}$$

7. Let A be the lighthouse, O, P the two positions of the vessel.

Then AP the direction of S., AO the direction of E.S.E., OP the direction of S.S.W.

$$\therefore \angle AOP = 90^\circ; \angle PAO = 90^\circ - 22\frac{1}{2}^\circ = 67\frac{1}{2}^\circ; \text{ and } AO = 4 \text{ miles.}$$

$$\therefore PO = AO \tan 67\frac{1}{2}^\circ = 4 \times 2.414 = 9.656 \text{ miles.}$$

$$\therefore \text{the vessel sails at the rate of } 9.656 \text{ miles per hour.}$$

8. We have $\angle CAB = 10^\circ + 50^\circ = 60^\circ$, $\angle ABC = 180^\circ - 50^\circ - 40^\circ = 90^\circ$,
 $BC = 10$ miles.

$$\therefore AB = BC \cot 60^\circ = \frac{10}{\sqrt{3}} = 5.77 \text{ mls.}; \quad AC = BC \operatorname{cosec} 60^\circ = \frac{20}{\sqrt{3}} = 11.54 \text{ mls.}$$

9. Let O be the lighthouse, and A, B the two positions of the ship.
 Then $\angle OAB = \angle OBA = 45^\circ$; $OA = OB = 15$ miles.

$$\therefore AB = 15\sqrt{2} \text{ miles} = \frac{15\sqrt{2} \times 60}{69} \text{ knots.}$$

$$\therefore \text{the ship in } 1\frac{1}{2} \text{ hours sails } \frac{15\sqrt{2} \times 60}{69} \text{ knots.}$$

$$\therefore \text{in a day it sails } \frac{15\sqrt{2} \times 60}{69} \times \frac{24 \times 2}{3} \text{ knots, that is, } 295.09 \text{ knots.}$$

10. Let O be the lighthouse, A, B the two positions of the coaster.
 Then AB is in direction S.E., and OA is in direction N.E.; $\therefore \angle OAB = 90^\circ$.
 Also $\angle AOB = 45^\circ + 15^\circ = 60^\circ$, and $OA = 9$ miles;

$$\therefore AB = OA \tan 60^\circ = 9\sqrt{3} \text{ miles}; \quad \therefore \text{coaster sails } 9\sqrt{3} \text{ miles in 3 hours.}$$

$$\therefore \text{rate of the coaster's sailing} = 5.196 \text{ miles per hour.}$$

$$\text{Also } OB = AO \sec 60^\circ = 18 \text{ miles.}$$

That is, the distance of the coaster from the lighthouse at time of second observation = 18 miles.

11. Let P be the position of the vessel when it is N.E. of A and N.W. of B . Then $\angle APB = 90^\circ$.

$$\text{Also } \angle PAB = 45^\circ - 15^\circ = 30^\circ. \quad \therefore PA = AB \cos 30^\circ = 6\sqrt{3} \text{ miles.}$$

Now the direction S. 15° E. is at right angles to the direction E. 15° N.

$\therefore$ the ship crosses AB at right angles. Draw PN perpendicular to AB .

Then $PN = AP \sin 30^\circ = 3\sqrt{3}$ miles; therefore the ship will reach N in $\frac{3\sqrt{3}}{10}$ hours, that is in 31.176 minutes.

$\therefore$ the ship will cross the line at about 31' past midnight.

12. Let P, Q be the two spires.

$$\text{Then } \angle PAB = 90^\circ, \text{ and } \angle PBQ = 37\frac{1}{2}^\circ - 7\frac{1}{2}^\circ = 30^\circ;$$

$$\therefore \angle QBA = 90^\circ - 37\frac{1}{2}^\circ - 22\frac{1}{2}^\circ = 30^\circ;$$

$$\therefore \angle BPQ = 30^\circ = \angle PBQ; \quad \text{so that } QB = QP = 1.5 \text{ miles.}$$

$$\therefore AB = BQ \cos 30^\circ = \frac{3\sqrt{3}}{4} \text{ miles.}$$

$$\therefore \text{the train travels } \frac{3\sqrt{3}}{4} \text{ miles in 2 minutes,}$$

that is, $\frac{3\sqrt{3}}{4} \times 30$ miles per hour, or 38.97 miles per hour.

EXAMPLES. VI. c. PAGE 48 A.

1. $h = 83 \tan 23^\circ 44' = 83 \times .4397$ yards
 $= 109$ ft., approximately.
2. $h = 173 \tan 63^\circ = 173 \times 1.9626$ ft.
 $= 339.53$ ft.
3. $h = 200 \sin 54^\circ = 200 \times .8090$ metres
 $= 161.8$ metres.
4. $d = 500 \sin 23^\circ = 500 \times .3907$ ft.
 $= 586.05$ ft.
5. The distance between two consecutive posts $= \frac{1760}{22} = 80$ yds.
 Then required distance $= 80 \tan 16^\circ 42' = 80 \times .3000$ yds.
 $= 24$ yds.

6. Let $ABCD$ be the square, and let the line be drawn from B to E , the middle point of AD .

Then $\tan ABE = \frac{AE}{AB} = .5$, whence $\angle ABE = 26^\circ 34'$;
 $\therefore \angle EBC = 90^\circ - 26^\circ 34' = 63^\circ 26'$.

7. Let D be the middle point of the base BC of the isosceles $\triangle ABC$, in which $AB = 3BC$.

Then $\cos DBA = \frac{BD}{BA} = \frac{1}{6} = .1666$,

whence $\angle B = 80^\circ 25' = \angle C$; $\therefore A = 19^\circ 10'$.

8. With the figure on p. 41, $QR = 160$ ft., $\angle PRT = 45^\circ$, $\angle PQT = 21^\circ 48'$; also $PT = RT$. If h is the required height in feet

$$\frac{h}{h+160} = \tan 21^\circ 48' = .4000;$$

$$\therefore h = .4h + 64; \therefore .6h = 64, \text{ and } h = 107.$$

9. With the same figure as in Ex. 8, $QR = 100$ yds., $\angle PRT = 54^\circ 24'$, $\angle PQT = 27^\circ 12'$. Let h be the height in feet; then

$$\frac{300 + RT}{h} = \cot 27^\circ 12' = \tan 62^\circ 48' = 1.9458;$$

and $RT = h \cot 54^\circ 24' = h \tan 35^\circ 36' = h \times .7159;$

$$\therefore 300 + h \times .7159 = h \times 1.9458;$$

that is, $1.2299h = 300$; whence $h = 244$, nearly.

Or thus: Since $\angle RPQ = 27^\circ 12'$, $\therefore PR = QR = 100$ yds;

$$\therefore h = 300 \sin 54^\circ 24' = 300 \times .8131 = 244.$$

10. With the same figure and notation as in Ex. 9,

$$\frac{1760 + RT}{h} = \tan 73^\circ 18' = 3.3332;$$

and

$$RT = h \tan 35^\circ = h \times .7002;$$

$$\therefore 1760 + h \times .7002 = h \times 3.3332;$$

that is, $2.6330h = 1760$; whence $h = 668$, nearly.

11. Let AB represent the top, and DE the bottom of the trench. Draw AC and BF perp. to ED and DE produced.

Then $CD = 8 \tan 12^\circ = 8 \times .2126 = 1.7008 \text{ ft.};$

whence $CE = 10.7008 \text{ ft.} \therefore EF = 4.2992 \text{ ft.}$

In $\triangle EBF$, $\tan B = \frac{4.2992}{8} = .5374;$

whence $B = 28^\circ 15'.$

12. Let C and B be the first and second positions of the observer; then $\angle ACB = 90^\circ$, and $\angle ADB = 143^\circ 24'$. $\therefore \angle ADC = 36^\circ 36'.$

Now $AC = 630 \tan 36^\circ 36' = 630 \times .7427 = 467.9 \text{ m.}$

$$AD = \frac{63}{\cos 36^\circ 36'} = \frac{63}{.8028} = 784.7 \text{ m.}$$

13. Let A be the point of observation, B the top, and C the bottom of the tower. Draw AE horizontally to meet BC in E . Then $\angle EAC = 17^\circ$, and $EC = AD = 30 \text{ ft.}$

$$AE = EC \cot 17^\circ = 30 \tan 73^\circ = (30 \times 3.2709) \text{ ft.} \\ = 98.127 \text{ ft.}$$

Again $BE = AE \tan 42^\circ = (98.127 \times .9004) \text{ ft.} \\ = 88.3506 \text{ ft.};$

$$\therefore \text{required height} = (30 + 88.3506) \text{ ft.} \\ = 118.35 \text{ ft.}$$

14. See fig. on page 44. Let $OA = x$, $AB = y$;

then $y = x \cot 35^\circ = x \tan 55^\circ = x \times 1.4281,$

$$\frac{700 + y}{x} = \cot 14^\circ = \tan 76^\circ = 4.0108;$$

$$\therefore 700 + x \times 1.4281 = x \times 4.0108;$$

that is, $700 = x \times 2.5827$; whence $x = 271.$

15. Let O be the lighthouse, and A, B the two positions of the ship. Then $\angle BOA = 90^\circ$, $\angle OBA = 32^\circ$, $AB = 15 \text{ mi.}$

Now $OA = AB \sin ABO = 15 \sin 32^\circ \\ = 15 \times .5299 = 7.9485 \text{ mi.}$

16. Let O be the house, and A, B the two positions. Then $\angle BOA = 90^\circ$, $\angle OBA = 52^\circ$, $AB = 2$ km. Let OC be perp. to AB ;

then $OB = 2000 \cos 52^\circ = 2000 \times .6157 = 1231.4$ m.

$$OC = OB \sin 52^\circ = 1231.4 \times .7880 = 970.3 \text{ m.}$$

17. Let L be the lighthouse, and S_1, S_2 the two positions of the ship. Then $\angle S_1OS_2 = 90^\circ$, $\angle S_2S_1L = 56^\circ$, $LS_1 = 12$ mi.

Now $S_1S_2 = \frac{12}{\cos 56^\circ} = \frac{12}{.5592}$ mi.; and since the ship has been sailing $\frac{7}{6}$

of an hour, the number of miles per day $= \frac{12}{.5592} \times \frac{6}{7} \times 24 = 441.5$.

18. Let B be the battery, and S_1, S_2 the two positions of the ship. Draw S_1C perp. to BS_2 ; then $\angle CS_1B = 45^\circ$, $S_1B = 2.5$ mi., and $BS_2 = 4$ mi.

Now $S_1C = 2.5 \sin 45^\circ = 2.5 \times .7071 = 1.7678$ mi.

$$S_2C = 4 - 1.7678 = 2.2322 \text{ mi.}$$

$$\tan S_1S_2C = \frac{1.7678}{2.2322} = .7912; \text{ whence } \angle S_1S_2C = 38^\circ 23'.$$

$\therefore S_2$ lies $38^\circ 23'$ E. of N. from S_1 .

19. See fig. on page 47.

Here $\angle BAE = 49^\circ$, $\angle EAC = 41^\circ$; $\therefore \angle BAC = 90^\circ$.

Also $\angle ACN' = 90^\circ - 41^\circ = 49^\circ$, and $\angle BCN' = 15^\circ$;

$$\therefore \angle ACB = 49^\circ - 15^\circ = 34^\circ.$$

Now $AB = AC \tan 34^\circ = 20 \times .6745 = 13.49$ mi.

$$BC = \frac{AC}{\cos 34^\circ} = \frac{20}{.8290} = 24.12 \text{ mi.}$$

EXAMPLES. VII. a. PAGE 54.

For Examples 1—22, see Art. 64; the following solutions will suffice as illustrations.

$$6. \text{ Radian measure of } 57\frac{1}{2} \text{ degrees} = \frac{57\frac{1}{2}}{180} \pi = \frac{23\pi}{72}.$$

$$7. \text{ Radian measure of } 14\frac{2}{5} \text{ degrees} = \frac{14\frac{2}{5}}{180} \pi = \frac{2\pi}{25}.$$

$$10. \text{ Radian measure of } 37\frac{1}{2} \text{ degrees} = \frac{37\frac{1}{2}}{180} \times 3.1416 = .6545.$$

11. Radian measure of $68\frac{3}{4}$ degrees $= \frac{68\frac{3}{4}}{180} \times 3.1416$
 $= \frac{275}{4 \times 180} \times 3.1416 = \frac{55}{4 \times 3} \times .2618 = 1.1999.$
16. $\frac{7\pi}{45}$ radians $= \frac{7 \times 180}{45}$ degrees $= 28^\circ.$
19. $.3927$ radians $= .3927 \times \frac{180^\circ}{\pi}$ [Art. 63]
 $= \frac{3927}{31416} \times 180^\circ = \frac{180^\circ}{8} = 22^\circ 30'.$
22. 2.8798 radians $= 2.8798 \times \frac{180^\circ}{\pi} = \frac{28798}{31416} \times 180^\circ = \frac{11}{12} \times 180^\circ = 165^\circ.$
23. Here $\frac{\theta}{\pi} = \frac{36.54}{180} = \frac{2.03}{10};$
 $\therefore \theta = .203 \times \frac{22}{7} = .638.$

$$\begin{array}{r} 60 \overline{) 24} \\ 60 \overline{) 32.4} \\ \hline .64 \end{array}$$
25. Here $\frac{\theta}{\pi} = \frac{116.046}{180} = .6447;$
 $\therefore \theta = .6447 \times \frac{22}{7} = 2.0262.$

$$\begin{array}{r} 60 \overline{) 45.6} \\ 60 \overline{) 276} \\ \hline .046 \end{array}$$
27. A radian $= \frac{180}{\pi}$ degrees;
 $\therefore$ no. of seconds in a radian $= \frac{180 \times 60 \times 60}{\pi}$
 $= 180 \times 60 \times 60 \times .31831$
 $= 206265$ nearly.
28. Since $1^\circ = \frac{\pi}{180}$ radians, the radian measure of $1''$
 $= \frac{3.1416}{180 \times 60 \times 60} = .0000048.$

EXAMPLES. VII. b. PAGE 56.

5. $\cot^2 \frac{\pi}{6} + 4 \cos^2 \frac{\pi}{4} + 3 \sec^2 \frac{\pi}{6} = (\sqrt{3})^2 + 4 \left(\frac{1}{\sqrt{2}} \right)^2 + 3 \left(\frac{2}{\sqrt{3}} \right)^2 = 3 + 2 + 4 = 9.$
6. $3 \tan^2 \frac{\pi}{6} - \frac{1}{3} \sin^2 \frac{\pi}{3} - \frac{1}{2} \operatorname{cosec}^2 \frac{\pi}{4} + \frac{4}{3} \cos^2 \frac{\pi}{6}$
 $= 3 \left(\frac{1}{\sqrt{3}} \right)^2 - \frac{1}{3} \left(\frac{\sqrt{3}}{2} \right)^2 - \frac{1}{2} (\sqrt{2})^2 + \frac{4}{3} \left(\frac{\sqrt{3}}{2} \right)^2$
 $= 1 - \frac{1}{4} - 1 + 1 = \frac{3}{4}.$

7. $\left(\sin \frac{\pi}{6} + \cos \frac{\pi}{6}\right) \left(\sin \frac{\pi}{3} - \cos \frac{\pi}{3}\right) \sec \frac{\pi}{3}$
 $= \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) 2 = 2 \left(\frac{3}{4} - \frac{1}{4}\right) = 1.$
8. First side $= \sin \theta \operatorname{cosec} \theta - \cot \theta \tan \theta = 1 - 1 = 0.$
9. First side $= \frac{\cos^2 \theta}{\sin \theta} - \cot^2 \theta \sin \theta = \frac{\cos^2 \theta}{\sin \theta} - \frac{\cos^2 \theta}{\sin \theta} = 0.$
10. First side $= \frac{\cos^2 \theta}{\operatorname{cosec} \theta} \cdot \frac{\sec \theta}{\tan \theta} = \cos^2 \theta \sin \theta \times \frac{1}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta} = \cos^2 \theta.$
11. First side $= \tan \theta + \cot \theta = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}$
 $= \frac{\cos^2 \theta + \sin^2 \theta}{\cos \theta \sin \theta} = \frac{1}{\cos \theta \sin \theta} = \sec \theta \operatorname{cosec} \theta = \sec \theta \sec \left(\frac{\pi}{2} - \theta\right).$
12. First side $= \sec^2 \theta + \operatorname{cosec}^2 \theta = \sec^2 \theta (1 + \cot^2 \theta)$
 $= \sec^2 \theta \operatorname{cosec}^2 \theta = (1 + \tan^2 \theta) \sec^2 \left(\frac{\pi}{2} - \theta\right).$
14. See Art. 69.
15. The second side $= \frac{1}{\cos^2 \frac{\pi}{3}} - \frac{1}{\cos^2 \frac{\pi}{6}} = 4 - \frac{4}{3} = \frac{8}{3}$
 $= 3 - \frac{1}{3} = \tan^2 \frac{\pi}{3} - \cot^2 \frac{\pi}{3}.$
16. The expression $= (\sin \theta + \cos \theta)^2 + (\cos \theta - \sin \theta)^2$
 $= 2 (\sin^2 \theta + \cos^2 \theta) = 2.$

EXAMPLES. VII. c. PAGE 60.

1. Here $\frac{\text{arc}}{\text{radius}} = \frac{1.6}{8} = \frac{1}{5} = \text{radian measure required.}$
2. Here $r = \frac{a}{\theta} = \frac{219}{.73} = 300 \text{ ft.}$
3. Radian measure $= \frac{7.5}{7.5} = 1.$
4. Here $a = r\theta = 1.625 \times 3.6 = 5.85 \text{ yds.}$
5. Here $a = 627 \text{ inches; } \theta = 1.9. \therefore r = \frac{a}{\theta} = \frac{627}{1.9} = 330 \text{ inches.}$
6. Each revolution $= 2\pi \text{ radians;}$
 $\therefore 5 \text{ radians} = \frac{5}{2\pi} = \frac{7 \times 5}{44} \text{ revolutions.}$

And each revolution takes $\frac{1}{35}$ of a second;

$$\therefore \text{required time} = \frac{1}{35} \times \frac{35}{44} = \frac{1}{44} \text{ of a second.}$$

7. Here $a = r\theta$, where $r = 28$ inches and $\theta = \frac{1}{3} \times \frac{44}{7}$.

$$\therefore a = \frac{44}{21} \times 28 = 58\frac{2}{3} \text{ inches.}$$

8. Radian measure of $75^\circ = \frac{75 \times 3.1416}{180}$.

If r be the length of the rope in yards, we have

$$r = \frac{52.36 \times 180}{3.1416 \times 75} = 40 \text{ yards.}$$

9. Here $a = r\theta$, where $r = 3960$ miles, and $\theta =$ radian measure of 1 minute;

$$\therefore a = 3960 \times \frac{3.1416}{180 \times 60} = 1.15192 \text{ miles.}$$

10. The number of radians in the angle $= \frac{11}{2} \times \frac{1}{1760 \times 12 \times 3}$;

$$\therefore \text{number of seconds} = \frac{11}{2} \times \frac{1}{1760 \times 12 \times 3} \times \frac{180}{3.1416} \times 60 \times 60$$

$$= 17.904, \text{ on reduction.}$$

11. With the figure on p. 59, we have to find the angle POQ when $PO = 3960$ miles, and $PQ = 145.2$.

$$\therefore \text{radian measure} = \frac{145.2}{3960};$$

$$\therefore \text{no. of degrees} = \frac{145.2}{3960} \times \frac{180 \times 7}{22} = \frac{1452}{220} \times \frac{7}{22} = \frac{66 \times 7}{220} = \frac{21}{10}.$$

$$\therefore \text{the angle} = 2^\circ 6'.$$

12. Here $r = \frac{a}{\theta}$, where $a = 1$ foot, and θ is the radian measure of $\frac{14}{11}$ degrees.

$$\therefore r = 1 \div \left(\frac{14}{11} \times \frac{\pi}{180} \right) = \frac{11}{14} \times \frac{180 \times 7}{22} = 45 \text{ ft.}$$

MISCELLANEOUS EXAMPLES. B. PAGE 61.

1. $D = \frac{180}{\pi} \times .15708 = \frac{180 \times 15708}{314160} = 9.$

2. See figure on p. 14. $b = c \cos A = \frac{110\sqrt{3}}{2} = 55 \times 1.732 = 95.26.$

3. If $\frac{12\pi}{23}$ is represented by $\frac{8}{5}$,

$$\pi \dots\dots\dots \frac{8}{5} \times \frac{25}{12}, \text{ or } \frac{10}{3}.$$

$\therefore 180 \div \frac{10}{3}$ is the number of degrees in the unit angle; that is, the unit is 54° .

4. Here $r = \frac{a}{\theta}$, where $a = 1$ inch and θ is the radian measure of $1'$.

$$\therefore r = 1 \div \left(\frac{\pi}{180} \times \frac{1}{60} \right) = 180 \times 60 \times \frac{1}{\pi} = 180 \times 60 \times .31831 = 3438 \text{ inches.}$$

$$\begin{aligned} 5. \quad (1) \quad \text{First side} &= (\sin a + \cos a) \left(\frac{\sin a}{\cos a} + \frac{\cos a}{\sin a} \right) \\ &= \frac{(\sin a + \cos a) (\sin^2 a + \cos^2 a)}{\sin a \cos a} = \sec a + \operatorname{cosec} a. \end{aligned}$$

$$\begin{aligned} (2) \quad \text{First side} &= (\sqrt{3} + 1) (3 - \sqrt{3}) = \sqrt{3} (\sqrt{3} + 1) (\sqrt{3} - 1) \\ &= 3\sqrt{3} - \sqrt{3} = \tan^3 60^\circ - 2 \sin 60^\circ. \end{aligned}$$

6. In the figure on p. 14, if BC represents the chimney and AC the shadow we have

$$\tan A = \frac{60}{3 \times 20\sqrt{3}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}; \quad \therefore A = 30^\circ.$$

$$\begin{aligned} 7. \quad (1) \quad \text{First side} &= 2 \tan^2 \theta + 5 \tan \theta + 2 \\ &= 2 (1 + \tan^2 \theta) + 5 \tan \theta = 2 \sec^2 \theta + 5 \tan \theta. \end{aligned}$$

$$(2) \quad \frac{\cot^2 a}{1 + \operatorname{cosec} a} = \frac{\operatorname{cosec}^2 a - 1}{\operatorname{cosec} a + 1} = \operatorname{cosec} a - 1.$$

8. Expressed in radians the third angle $= \pi - \left(\frac{\pi}{4} + \frac{5\pi}{8} \right) = \frac{\pi}{8}$. The sexagesimal equivalent is $22\frac{1}{2}^\circ$.

9. Let x be the number of degrees in the angle; then

$$x = 14 \left(\frac{\pi}{180} x \right) + 51, \text{ or } x - \frac{11}{45} x = 51; \text{ whence } x = 67\frac{1}{2}.$$

10. With the figure of Art. 45, we have

$$a = b \tan 60^\circ = 6\sqrt{3}; \quad c = b \sec 60^\circ = 6 \times 2 = 12.$$

Also the perpendicular from C on $AB = b \sin 60^\circ = 3\sqrt{3}$.

11. (1) First side $= \cot \theta + \tan \theta = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}$
 $= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1}{\sin \theta \cos \theta} = \operatorname{cosec} \theta \sec \theta = \operatorname{cosec} \theta \operatorname{cosec} \left(\frac{\pi}{2} - \theta \right).$
 (2) First side $= \operatorname{cosec}^2 \theta + \sec^2 \theta = \sec^2 \theta \operatorname{cosec}^2 \theta$ [Art. 31, Ex. 1.]
 $= \operatorname{cosec}^2 \theta \operatorname{cosec}^2 \left(\frac{\pi}{2} - \theta \right).$

12. In the figure on p. 41, let PT be the pillar; then
 $QR = 20$ ft., $\angle PQR = 30^\circ$, $\angle PRT = 60^\circ$, and $PR = QR = 20$ ft.
 $\therefore PT = PR \sin 60^\circ = \frac{20\sqrt{3}}{2} = 10\sqrt{3} = 17.32$ ft.

13. First side $= \sin^2 A \left(\frac{1}{\cos^2 A} - 1 \right) = \frac{\sin^2 A (1 - \cos^2 A)}{\cos^2 A}$
 $= \sin^2 A \cdot \sin^2 A \cdot \sec^2 A = \sin^4 A \sec^2 A.$

14. x grades $= \frac{9x}{10}$ degrees;
 also $\frac{\pi x}{300}$ radians $= \frac{\pi x}{300} \times \frac{180}{\pi} = \frac{3x}{5}$ degrees;

$\therefore 3x + \frac{9x}{10} + \frac{3x}{5} = 180$; whence $x = 40$. Thus the angles are 120° , 36° , 24° .

15. Expression $= \left(\frac{\sqrt{3}}{2} \right)^3 \sqrt{3} - 2(\sqrt{2})^2 + 3 \cdot \frac{1}{2} \cdot 1 - (\sqrt{3})^2$
 $= \frac{9}{8} - 4 + \frac{3}{2} - 3 = -\frac{35}{8}.$

16. (1) First side $= 1 + \tan^2 A + 2 \tan A + 1 + \cot^2 A + 2 \cot A$
 $= \sec^2 A + \operatorname{cosec}^2 A + 2 \cdot \frac{\sin^2 A + \cos^2 A}{\sin A \cos A}$
 $= \sec^2 A + \operatorname{cosec}^2 A + 2 \sec A \operatorname{cosec} A = (\sec A + \operatorname{cosec} A)^2.$
 (2) First side $= (\sec a - 1)^2 - \sin^2 a (\sec a - 1)^2 = (\sec a - 1)^2 (1 - \sin^2 a)$
 $= (\sec a - 1)^2 \cos^2 a = (1 - \cos a)^2.$

17. (1) $a^2 + b^2 > 2ab$, since $(a - b)^2$ is positive.
 Therefore $\frac{a^2 + b^2}{2ab} > 1$. Hence $\operatorname{cosec} \theta = \frac{a^2 + b^2}{2ab}$ is possible.

- (2) $a^2 + 1 > 2a$: so that $a + \frac{1}{a} > 2$.

Hence $2 \sin \theta = a + \frac{1}{a}$ is impossible [Art. 16].

18. The height of the balloon = $660 \tan 60^\circ \text{ ft.} = 660\sqrt{3} \text{ feet.}$

$\therefore$ the balloon rises $660 \times \sqrt{3} \text{ feet in } 1.5 \text{ minutes.}$

$\therefore$ it rises $\frac{660 \times \sqrt{3} \times 60}{1.5 \times 3 \times 1760}$, or $8.66 \text{ miles per hour.}$

19. Let x° be the common difference between the angles,
then they are $36^\circ, 36^\circ + x^\circ, 36^\circ + 2x^\circ,$

$$\therefore 3x + 3 \times 36 = 180; \text{ whence } x = 24.$$

$\therefore$ the angles are $36^\circ, 60^\circ, 84^\circ$, or $\frac{\pi}{5}, \frac{\pi}{3}, \frac{7\pi}{15}$ radians.

20. First side = $\sin^2 \alpha (1 + \tan^2 \beta) + \tan^2 \beta (1 - \sin^2 \alpha) = \sin^2 \alpha + \tan^2 \beta.$

21. Let $CD = x$ = the perpendicular.

Then $\angle CBD = 180^\circ - 116^\circ 33' = 63^\circ 27'$; $\therefore x = DB \tan 63^\circ 27' = 2DB.$

And $x = DA \tan 42^\circ = \left(\frac{x}{2} + 55\right) \times .9$; whence $x = 4.5 \times 20 = 90.$

22. (1) First side = $\frac{\cos \alpha (1 + \cos \alpha) + \sin^2 \alpha}{\sin \alpha (1 + \cos \alpha)} = \frac{\cos \alpha + 1}{\sin \alpha (1 + \cos \alpha)} = \text{cosec } \alpha.$

(2) First side = $\frac{1 - \cos \alpha}{\sin \alpha} \left(\frac{1}{\cos \alpha} - \cos \alpha \right) = (1 - \cos \alpha) \frac{\sin \alpha}{\cos \alpha} = \tan \alpha - \sin \alpha.$

23. First side = $\left(\frac{1 + \sqrt{3}}{1 - \sqrt{3}} \right)^2 = \frac{2 + \sqrt{3}}{2 - \sqrt{3}} = \frac{1 + \frac{\sqrt{3}}{2}}{1 - \frac{\sqrt{3}}{2}} = \frac{1 + \cos 30^\circ}{1 - \cos 30^\circ}.$

24. Let the man start from A and walk to B , and let C denote the position of the windmill.

Then we have $\angle ACB = 90^\circ, \angle CAB = 30^\circ, BC = 1 \text{ mile.}$

$\therefore AB = BC \text{ cosec } 30^\circ = 2 \text{ miles. } AC = BC \tan 60^\circ = 1.732 \text{ miles.}$

And rate of walking is 2 miles per half hour, or 4 miles an hour.

25. The complement of $\frac{3\pi}{8} = \frac{\pi}{2} - \frac{3\pi}{8} = \frac{\pi}{8}$ radians.

26. (1) $3 \sin \theta + 4 - 4 \sin^2 \theta = \frac{9}{2},$ (2) $\tan \theta + \frac{2}{\sqrt{3}} = \frac{1}{\tan \theta},$
 $8 \sin^2 \theta - 6 \sin \theta + 1 = 0; \quad \sqrt{3} \tan^2 \theta + 2 \tan \theta - \sqrt{3} = 0;$
 $\therefore (4 \sin \theta - 1)(2 \sin \theta - 1) = 0; \quad \therefore (\sqrt{3} \tan \theta - 1)(\tan \theta + \sqrt{3}) = 0;$
 $\therefore \sin \theta = \frac{1}{4}, \text{ or } \frac{1}{2}, \quad \therefore \tan \theta = \frac{1}{\sqrt{3}}, \text{ or } -\sqrt{3}.$
 $\therefore \theta = 30^\circ, \text{ or } 14^\circ 29'. \quad \therefore \theta = 30^\circ.$

$$27. \text{ Here } \frac{5 \sin a - 3 \cos a}{\sin a + 2 \cos a} = \frac{5 \tan a - 3}{\tan a + 2} = \frac{20 - 15}{4 + 10} = \frac{5}{14}.$$

$$\begin{aligned} 28. \text{ First side} &= \frac{1 - \sin A \cos A}{\cos A \cdot \frac{\sin A - \cos A}{\sin A \cos A}} \times \frac{\sin A - \cos A}{\sin^2 A - \sin A \cos A + \cos^2 A} \\ &= \frac{\sin A (1 - \sin A \cos A)}{1 - \sin A \cos A} = \sin A. \end{aligned}$$

$$29. \text{ The distance} = 195 \cdot 2 \operatorname{cosec} 77^\circ 26' \text{ yds.} = \frac{195 \cdot 2}{.976} = 200 \text{ yds.}$$

$$30. \text{ We have } 70^\circ = \frac{7\pi}{18} \text{ radians.}$$

$$\therefore \text{ distance required} = 27 \times \frac{7\pi}{18} \text{ feet} = \frac{21}{2} \times \frac{22}{7} = 33 \text{ feet.}$$

EXAMPLES. VIII. a. PAGE 70.

$$18. \sin 420^\circ = \sin (360^\circ + 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}.$$

$$20. \tan (-315^\circ) = \tan (-360^\circ + 45^\circ) = \tan 45^\circ = 1.$$

$$22. \operatorname{cosec} (-330^\circ) = \operatorname{cosec} (-360^\circ + 30^\circ) = \operatorname{cosec} 30^\circ = 2.$$

$$24. \cot \frac{17\pi}{4} = \cot \left(4\pi + \frac{\pi}{4} \right) = \cot \frac{\pi}{4} = 1.$$

$$26. \tan \left(-\frac{5\pi}{3} \right) = \tan \left(-2\pi + \frac{\pi}{3} \right) = \tan \frac{\pi}{3} = \sqrt{3}.$$

EXAMPLES. VIII. b. PAGE 72.

1. The boundary line of 120° lies in the second quadrant,

$$\therefore \cos 120^\circ = -\sqrt{1 - \sin^2 120^\circ} = -\sqrt{1 - \frac{3}{4}} = -\frac{1}{2};$$

$$\therefore \tan 120^\circ = \frac{\sin 120^\circ}{\cos 120^\circ} = -\sqrt{3}.$$

2. The boundary line of 135° lies in the second quadrant;

$$\therefore \sec 135^\circ = -\sqrt{1 + \tan^2 135^\circ} = -\sqrt{2};$$

and $\sin 135^\circ \sec 135^\circ = -1; \therefore \sin 135^\circ = \frac{1}{\sqrt{2}}.$

3. The boundary line of 240° lies in the third quadrant;

$$\therefore \sec 240^\circ = -\sqrt{1 + \tan^2 240^\circ} = -2; \quad \therefore \cos 240^\circ = -\frac{1}{2}.$$

4. The boundary line of $202^\circ 37'$ lies in the third quadrant;

$$\therefore \cos A = -\sqrt{1 - \sin^2 A} = -\sqrt{1 - \frac{25}{169}} = -\frac{12}{13}. \quad \text{Also } \cot A = \frac{\cos A}{\sin A} = \frac{12}{5}.$$

5. The boundary line of $143^\circ 8'$ lies in the second quadrant; and

$$\operatorname{cosec} A = 1\frac{2}{5}; \quad \therefore \sin A = \frac{3}{5}; \quad \therefore \cos A = -\sqrt{1 - \sin^2 A} = -\frac{4}{5};$$

$$\therefore \sec A = -\frac{5}{4}. \quad \text{Hence } \tan A = \frac{\sin A}{\cos A} = -\frac{3}{4}.$$

6. The boundary of $216^\circ 52'$ lies in the third quadrant;

$$\therefore \sin A = -\sqrt{1 - \cos^2 A} = -\sqrt{1 - \frac{16}{25}} = -\frac{3}{5}. \quad \text{Also } \cot A = \frac{\cos A}{\sin A} = \frac{4}{3}.$$

7. The boundary line of $\frac{2\pi}{3}$ lies in the second quadrant;

$$\text{and } \sec \frac{2\pi}{3} = -2; \quad \therefore \cos \frac{2\pi}{3} = -\frac{1}{2}. \quad \therefore \sin \frac{2\pi}{3} = +\sqrt{1 - \cos^2 \frac{2\pi}{3}} = \frac{\sqrt{3}}{2}.$$

$$\cot \frac{2\pi}{3} = \frac{\cos \frac{2\pi}{3}}{\sin \frac{2\pi}{3}} = -\frac{1}{\sqrt{3}}.$$

8. The boundary line of $\frac{5\pi}{4}$ lies in the third quadrant;

$$\therefore \cos \frac{5\pi}{4} = -\sqrt{1 - \sin^2 \frac{5\pi}{4}} = -\frac{1}{\sqrt{2}};$$

$$\therefore \sec \frac{5\pi}{4} = -\sqrt{2}, \quad \text{and } \tan \frac{5\pi}{4} = \frac{\sin \frac{5\pi}{4}}{\cos \frac{5\pi}{4}} = 1.$$

9. We have $\sin A = \pm \sqrt{1 - \cos^2 A} = \pm \sqrt{1 - \frac{144}{169}} = \pm \frac{5}{13}.$

$$\therefore \tan A = \frac{\sin A}{\cos A} = \pm \frac{5}{12}.$$

EXAMPLES. IX. PAGE 79.

6. Expression = $1 \cdot (-1)^2 - 2(-1) \cdot 1 = 1 + 2 = 3$.
 7. Expression = $3 \cdot 0 \cdot (-1) + 2 \cdot 1 - 1 = 2 - 1 = 1$.
 8. Expression = $2 \cdot (-1)^2 \cdot 1 + 3(-1)^3 - 1 = 2 - 3 - 1 = -2$.
 9. Expression = $0 \times 0 + 1 - (-1) = 1 + 1 = 2$.

MISCELLANEOUS EXAMPLES. C. PAGE 80.

1. If $\tan A = -\frac{3}{4}$, the boundary of A will lie either in second or fourth quadrant. [See figure on page 72.]

In either position the radius vector = $\sqrt{3^2 + 4^2} = 5$.

Hence $\cos XOP = -\frac{4}{5}; \quad \cos XOP' = \frac{4}{5}$.

2. First side = $(2 + \sin A)(1 - 2 \sin A) \sec A$
 $= (2 - 3 \sin A - 2 \sin^2 A) \sec A$
 $= (2 \cos^2 A - 3 \sin A) \sec A = 2 \cos A - 3 \tan A$.

3. $a = \sqrt{c^2 - b^2} = \sqrt{(21)^2 - (10 \cdot 5)^2} = 21 \sqrt{1 - \frac{1}{4}} = \frac{21\sqrt{3}}{2}$.

$\sin A = \frac{a}{c} = \frac{\sqrt{3}}{2}; \quad \text{whence } A = 60^\circ, B = 30^\circ$.

4. A lies between 180° and 270° ;

$\therefore \tan A = +\sqrt{\sec^2 A - 1} = \sqrt{\left(\frac{25}{7}\right)^2 - 1} = \frac{24}{7}; \quad \text{and } \cot A = \frac{7}{24}$.

5. We have $19^\circ = \frac{19\pi}{180}$ radians.

Also the radius of the earth = 3960 miles.

$\therefore$ required distance = $\frac{19\pi}{180} \times 3960 = 19 \times 22 \times \pi = 1313$ miles nearly.

6. Let AB represent the cliff and P, Q the positions of the two boats.
 Then $AB = 200$ ft., $\angle APB = 34^\circ 30'$, $\angle AQB = 18^\circ 40'$;

$\therefore QB = AB \cot 18^\circ 40' = 200 \times 2 \cdot 96 = 592$ ft.

$PB = AB \cot 34^\circ 30' = 200 \times 1 \cdot 455 = 291$ ft.

$\therefore$ required distance = $QB - PB = 301$ ft.

7. Since the boundary line of A is in the third quadrant,

$$\therefore \sec A = -\sqrt{1 + \tan^2 A} = -\sqrt{1 + \frac{16}{9}} = -\frac{5}{3},$$

$$\therefore \cos A = -\frac{3}{5}, \quad \text{and} \quad \sin A = -\frac{4}{5}.$$

$$\therefore 2 \cot A - 5 \cos A + \sin A = 2 \cdot \frac{3}{4} - 5 \left(-\frac{3}{5} \right) - \frac{4}{5} = \frac{3}{2} + 3 - \frac{4}{5} = 3\frac{7}{10}.$$

8. We have $71^\circ 36' 3\cdot6'' = 71\cdot601^\circ = \frac{71\cdot601\pi}{180}$ radians.

$$\therefore \text{required radius} = 15 \div \frac{71\cdot601\pi}{180} = \frac{15 \times 180}{71\cdot601\pi} = 12\cdot003 \text{ inches.}$$

$$\begin{aligned} 9. \quad \text{First side} &= \frac{\tan^2 \theta}{\sec^2 \theta} + \frac{\cot^2 \theta}{\operatorname{cosec}^2 \theta} \\ &= \frac{\sin^3 \theta}{\cos \theta} + \frac{\cos^3 \theta}{\sin \theta} = \frac{\sin^4 \theta + \cos^4 \theta}{\sin \theta \cos \theta} \\ &= \frac{(\sin^2 \theta + \cos^2 \theta)^2 - 2 \sin^2 \theta \cos^2 \theta}{\sin \theta \cos \theta} = \frac{1 - 2 \sin^2 \theta \cos^2 \theta}{\sin \theta \cos \theta}. \end{aligned}$$

10. Let AB represent the flagstaff, BC the tower, and let D be the position of the observer.

$$\text{Then } \angle BDC = 68^\circ 11', \quad \angle ADB = 2^\circ 10'; \quad \therefore \angle ADC = 70^\circ 21'.$$

$$\text{Let } BC = x \text{ ft., then } x + 24 = DO \tan 70^\circ 21', \text{ and } DC = x \cot 68^\circ 11'.$$

$$\therefore x + 24 = x \cot 68^\circ 11' \tan 70^\circ 21'; \quad \therefore x + 24 = x \times 2\cdot8 \times \cdot4 = 1\cdot12x;$$

whence $x = 200$; that is, the height of the tower is 200 ft.

11. If $\tan A = \cdot5$, and $\tan B = \cdot3333$, from the Tables we have $A = 26^\circ 34'$, $B = 18^\circ 26'$; $\therefore A + B = 45^\circ$.

$$12. \quad \text{We have} \quad (4 \tan \theta - 3)(3 \tan \theta + 4) = 0;$$

whence $\tan \theta = \cdot75$, or $-1\cdot3333$.

$$\therefore, \text{ from the Tables, } \theta = 36^\circ 52', \text{ or } 180^\circ - 53^\circ 8',$$

that is, $\theta = 36^\circ 52'$, or $126^\circ 52'$.

$$\begin{aligned} 13. \quad \text{We have} \quad \frac{3\cdot7}{r} &= \text{radian measure of } 21^\circ 12' \\ &= \frac{\pi}{180} \times 21\cdot2; \end{aligned}$$

$$\therefore 180 \times 3\cdot7 = r \times 3\cdot1416 \times 21\cdot2,$$

or $666 = r \times 66.602$, approx.

$\therefore$ radius = 10 in., to the nearest inch.

If d be the number of miles between the two places

$$\frac{d}{4000} = \frac{3.7}{10}; \text{ whence } d = 1480.$$

14. Let T be the tower, and O_1, O_2 the two points of observation. Then it is easily seen that

$$\angle TO_1O_2 = 30^\circ, \angle TO_2O_1 = 60^\circ;$$

$$\therefore \angle O_1TO_2 = 90^\circ; \text{ also } TO_2 = 2 \text{ km.}$$

Now $O_1T = O_2T \tan 60^\circ = 2 \times 1.732 = 3.464 \text{ km.}$

Also $O_1O_2 = O_2T \sec 60^\circ = 4 \text{ km.}$; and since he walks this distance in 40 min. his rate of walking is 6 km. per hour.

15. From the Tables, $\alpha + \beta = 51^\circ 33'$, $\alpha - \beta = 47^\circ 5'$,

$$\therefore \alpha = 49^\circ 19', \beta = 2^\circ 14'.$$

16. The expression
$$= \frac{10 - 6 \cot \alpha}{4 + 3 \cot \alpha} = \frac{10 - 4}{4 + 2} = 1.$$

17. Let F be the fort, and S_1, S_2 the two positions of the ship. Then it is easily seen that $\angle FS_2S_1 = 90^\circ$, $\angle S_2S_1F = 43^\circ$; also $S_1S_2 = 20 \text{ mi.}$

$$\therefore FS_2 = 20 \tan 43^\circ = 20 \times .9325 = 18.65 \text{ mi.}$$

$$FS_1 = \frac{20}{\cos 43^\circ} = \frac{20}{.7314} = 27.346 \text{ mi.}$$

EXAMPLES. X. a. PAGE 87.

$$1. \cos 135^\circ = \cos (180^\circ - 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}.$$

$$2. \sin 150^\circ = \sin (180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}.$$

$$3. \tan 240^\circ = \tan (180^\circ + 60^\circ) = \tan 60^\circ = \sqrt{3}.$$

$$4. \operatorname{cosec} 225^\circ = \operatorname{cosec} (180^\circ + 45^\circ) = -\operatorname{cosec} 45^\circ = -\sqrt{2}.$$

$$5. \sin (-120^\circ) = -\sin 120^\circ = -\sin (180^\circ - 60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}.$$

$$6. \cot (-135^\circ) = -\cot 135^\circ = -\cot (180^\circ - 45^\circ) = \cot 45^\circ = 1.$$

$$7. \cot 315^\circ = \cot (180^\circ + 135^\circ) = \cot 135^\circ = -1.$$

$$8. \cos (-240^\circ) = \cos 240^\circ = \cos (180^\circ + 60^\circ) = -\cos 60^\circ = -\frac{1}{2}.$$

9. $\sec(-300^\circ) = \sec 300^\circ = \sec(180^\circ + 120^\circ) = -\sec 120^\circ = -\sec(180^\circ - 60^\circ) = \sec 60^\circ = 2.$
10. $\tan \frac{3\pi}{4} = \tan \left(\pi - \frac{\pi}{4} \right) = -\tan \frac{\pi}{4} = -1.$
11. $\sin \frac{4\pi}{3} = \sin \left(\pi + \frac{\pi}{3} \right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$
12. $\sec \frac{2\pi}{3} = \sec \left(\pi - \frac{\pi}{3} \right) = -\sec \frac{\pi}{3} = -2.$
13. $\operatorname{cosec} \left(-\frac{\pi}{6} \right) = -\operatorname{cosec} \frac{\pi}{6} = -2.$
14. $\cos \left(-\frac{3\pi}{4} \right) = \cos \frac{3\pi}{4} = \cos \left(\pi - \frac{\pi}{4} \right) = -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}}.$
15. $\cot \left(-\frac{5\pi}{6} \right) = -\cot \frac{5\pi}{6} = -\cot \left(\pi - \frac{\pi}{6} \right) = \cot \frac{\pi}{6} = \sqrt{3}.$
16. $\cos(270^\circ + A) = \cos(180^\circ + \overline{90^\circ + A}) = -\cos(90^\circ + A) = \sin A.$
17. $\cot(270^\circ - A) = \cot(180^\circ + \overline{90^\circ - A}) = \cot(90^\circ - A) = \tan A.$
18. $\sin(A - 90^\circ) = -\sin(90^\circ - A) = -\cos A.$
19. $\sec(A - 180^\circ) = \sec(180^\circ - A) = -\sec A.$
20. $\sin(270^\circ - A) = \sin(180^\circ + \overline{90^\circ - A}) = -\sin(90^\circ - A) = -\cos A.$
21. $\cot(A - 90^\circ) = -\cot(90^\circ - A) = -\tan A.$
22. $\sin \left(\theta - \frac{\pi}{2} \right) = -\sin \left(\frac{\pi}{2} - \theta \right) = -\cos \theta.$
23. $\tan(\theta - \pi) = -\tan(\pi - \theta) = \tan \theta.$
24. $\sec \left(\frac{3\pi}{2} - \theta \right) = \sec \left(\pi + \frac{\pi}{2} - \theta \right) = -\sec \left(\frac{\pi}{2} - \theta \right) = -\operatorname{cosec} \theta.$
25. Expression $= \tan A \cos A \operatorname{cosec} A = 1.$
26. Expression $= -\sin A + \sin A - (-\sin A) - (-\sin A) = 2 \sin A.$
27. Expression $= \sec^2 A - \tan^2 A = 1.$

EXAMPLES. X. b. PAGE 91.

1. $\cos 480^\circ = \cos(360^\circ + 120^\circ) = \cos 120^\circ = -\frac{1}{2}.$
2. $\sin 960^\circ = \sin(3 \times 360^\circ - 120^\circ) = -\sin 120^\circ = -\frac{\sqrt{3}}{2}.$
3. $\cos 780^\circ = \cos(2 \times 360^\circ + 60^\circ) = \cos 60^\circ = \frac{1}{2}.$

4. $\sin (-870^\circ) = \sin (-2 \times 360^\circ - 150^\circ) = -\sin 150^\circ = -\frac{1}{2}.$
5. $\sec 900^\circ = \sec (2 \times 360^\circ + 180^\circ) = \sec 180^\circ = -1.$
6. $\tan (-855^\circ) = -\tan 855^\circ = -\tan (2 \times 360^\circ + 135^\circ)$
 $= -\tan 135^\circ = -\tan (180^\circ - 45^\circ) = \tan 45^\circ = 1.$
7. $\operatorname{cosec} (-660^\circ) = -\operatorname{cosec} 660^\circ = -\operatorname{cosec} (2 \times 360^\circ - 60^\circ) = \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}.$
8. $\cot 840^\circ = \cot (2 \times 360^\circ + 120^\circ) = \cot 120^\circ$
 $= \cot (180^\circ - 60^\circ) = -\cot 60^\circ = -\frac{1}{\sqrt{3}}.$
9. $\operatorname{cosec} (-765^\circ) = -\operatorname{cosec} 765^\circ$
 $= -\operatorname{cosec} (2 \times 360^\circ + 45^\circ) = -\operatorname{cosec} 45^\circ = -\sqrt{2}.$
10. $\cos 1125^\circ = \cos (3 \times 360^\circ + 45^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}.$
11. $\cot 990^\circ = \cot (3 \times 360^\circ - 90^\circ) = -\cot 90^\circ = 0.$
12. $\sin 855^\circ = \sin (2 \times 360^\circ + 135^\circ) = \sin 135^\circ = \sin (180^\circ - 45^\circ)$
 $= \sin 45^\circ = \frac{1}{\sqrt{2}}.$
13. $\sec 1305^\circ = \sec (4 \times 360^\circ - 135^\circ) = \sec 135^\circ = -\sec 45^\circ = -\sqrt{2}.$
14. $\cos 960^\circ = \cos (3 \times 360^\circ - 120^\circ) = \cos 120^\circ = -\cos 60^\circ = -\frac{1}{2}.$
15. $\sec (-1575^\circ) = \sec 1575^\circ = \sec (4 \times 360^\circ + 135^\circ)$
 $= \sec 135^\circ = -\sec 45^\circ = -\sqrt{2}.$
16. $\sin \frac{15\pi}{4} = \sin \left(4\pi - \frac{\pi}{4} \right) = -\sin \frac{\pi}{4} = -\frac{1}{\sqrt{2}}.$
17. $\cot \frac{23\pi}{4} = \cot \left(6\pi - \frac{\pi}{4} \right) = -\cot \frac{\pi}{4} = -1.$
18. $\sec \frac{7\pi}{3} = \sec \left(2\pi + \frac{\pi}{3} \right) = \sec \frac{\pi}{3} = 2.$
19. $\cot \frac{16\pi}{3} = \cot \left(6\pi - \frac{2\pi}{3} \right) = -\cot \frac{2\pi}{3} = \cot \frac{\pi}{3} = \frac{1}{\sqrt{3}}.$
20. $\sec \left(\frac{3\pi}{2} + \frac{\pi}{3} \right) = \sec \left(2\pi + \frac{\pi}{3} - \frac{\pi}{2} \right) = \sec \left(2\pi - \frac{\pi}{6} \right) = \sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}.$
21. $\cos \theta = \frac{\sqrt{3}}{2} = \cos 30^\circ; \therefore \theta = 30^\circ$ satisfies the equation.

And $\cos 30^\circ = \cos (360^\circ - 30^\circ) = \cos 330^\circ.$

There are no angles whose boundary lines are in the second and third

quadrants which satisfy the equation since the cosine in those quadrants is negative.

$\therefore$ the positive angles are $30^\circ, 330^\circ$.

And the negative angles are $-(360^\circ - 30^\circ), -(360^\circ - 330^\circ)$.

That is, the angles are $\pm 30^\circ, \pm 330^\circ$.

$$22. \quad \sin \theta = -\frac{1}{2} = \sin (180^\circ + 30^\circ) = \sin 210^\circ; \quad \therefore \theta = 210^\circ \text{ is a solution.}$$

Also $\sin (360^\circ - 30^\circ) = -\sin 30^\circ = -\frac{1}{2}; \quad \therefore \theta = 330^\circ$ is another solution.

Thus $210^\circ, 330^\circ$ are the positive angles.

The negative angles are $-(360^\circ - 210^\circ), -(360^\circ - 330^\circ);$

$\therefore$ the required angles are $210^\circ, 330^\circ, -150^\circ, -30^\circ$.

$$23. \quad \tan \theta = -\sqrt{3} = \tan (180^\circ - 60^\circ) = \tan 120^\circ; \quad \therefore \theta = 120^\circ \text{ is a solution.}$$

Also $\tan (360^\circ - 60^\circ) = -\tan 60^\circ = -\sqrt{3}; \quad \therefore \theta = 300^\circ$ is another solution.

Thus $120^\circ, 300^\circ$ are the positive angles.

The negative angles are $-(360^\circ - 120^\circ), -(360^\circ - 300^\circ);$

$\therefore$ the required angles are $120^\circ, 300^\circ, -240^\circ, -60^\circ$.

$$24. \quad \cot \theta = -1 = -\cot 45^\circ = \cot 135^\circ.$$

Also $\cot 135^\circ = \cot (180^\circ + 135^\circ) = \cot 315^\circ$.

$\therefore$ the positive angles which satisfy the equation are $135^\circ, 315^\circ$.

The negative angles are $-(360^\circ - 135^\circ), -(360^\circ - 315^\circ)$.

$\therefore$ the required angles are $135^\circ, 315^\circ, -45^\circ, -225^\circ$.

25. Let the radius vector OP start from the position OX and revolve in the positive direction till it reaches the position OP , such that $\angle POX = A$. Then let it revolve in the negative direction through an angle of 180° , reaching the position OP' .

Then POP' is a straight line, and $\angle XOP' = A - 180^\circ$.

Draw $PM, P'M'$ perpendicular to XX' . Then the $\Delta^s OPM, OP'M'$ are geometrically equal.

$$\text{Then } \sec(A - 180^\circ) = \frac{OP'}{OM'} = -\frac{OP}{OM} = -\sec A.$$

26. Proceed as in Art. 97. Let the radius vector first revolve from OX through the angle A to the position OP . Again, let it revolve from OP through 270° and then further through an angle A to the position OP' ; draw $PM, P'M'$ perpendiculars to XX' . Then from the equal $\Delta^s OPM, OP'M'$, we have

$$P'M' = -OM, \quad O'M' = PM;$$

$$\therefore \tan(270^\circ + A) = \frac{P'M'}{O'M'} = -\frac{OM}{PM} = -\cot A.$$

27. Let OP be determined as before, and then let the radius vector turn back in the negative direction through an angle 90° to the position OP' . Draw perpendiculars as before.

$$\text{Then } \cos(A - 90^\circ) = \frac{OM'}{OP'} = \frac{PM}{OP} = \sin A.$$

$$28. \text{ First side} = \tan A - \tan A - \tan A = -\tan A = \tan(360^\circ - A).$$

$$29. \text{ First side} = \frac{\sin A}{\tan A} \cdot \frac{\tan A}{-\cot A} \cdot \frac{\cos A}{-\sin A} = \sin A.$$

$$30. \text{ Expression} = \frac{-\sin A}{-\sin A} - \frac{-\cot A}{\cot A} + \frac{\cos A}{\cos A} = 1 + 1 + 1 = 3.$$

$$31. \text{ Expression} = \frac{\operatorname{cosec} A}{-\sec A} \cdot \frac{\cos A}{-\sin A} = \frac{\cos^2 A}{\sin^2 A} = \cot^2 A.$$

$$32. \text{ Expression} = \frac{-\sin A \cdot \sec A \cdot (-\tan A)}{\sec A (-\sin A) \tan A} = -1.$$

$$33. \text{ First side} = \sin\left(\pi - \frac{\pi}{2} - \theta\right) \sin\left(\frac{\pi}{2} + \pi - \theta\right) \cot\left(\pi + \frac{\pi}{2} + \theta\right) \\ = \sin\left(\frac{\pi}{2} - \theta\right) \sin\left(\frac{3\pi}{2} - \theta\right) \cot\left(\frac{\pi}{2} + \theta\right).$$

$$34. \sin \alpha = \sin \frac{11\pi}{4} = \sin\left(2\pi + \frac{3\pi}{4}\right) = \sin \frac{3\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}};$$

$$\cos \alpha = \cos \frac{3\pi}{4} = -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}}. \quad \text{Also } \tan \alpha = -1.$$

$$\therefore \text{ Expression} = \frac{1}{2} - \frac{1}{2} - 2 - 2 = -4.$$

EXAMPLES. XI. a. PAGE 97.

$$1. \sin(A + 45^\circ) = \sin A \cos 45^\circ + \cos A \sin 45^\circ = \frac{1}{\sqrt{2}}(\sin A + \cos A).$$

$$2. \cos(A + 45^\circ) = \cos A \cos 45^\circ - \sin A \sin 45^\circ = \frac{1}{\sqrt{2}}(\cos A - \sin A).$$

$$3. 2 \sin(30^\circ - A) = 2(\sin 30^\circ \cos A - \cos 30^\circ \sin A) = \cos A - \sqrt{3} \sin A.$$

$$4. \cos A = \frac{4}{5}; \quad \therefore \sin A = \frac{3}{5}. \quad \cos B = \frac{3}{5}; \quad \therefore \sin B = \frac{4}{5}.$$

$$\therefore \sin(A + B) = \sin A \cos B + \cos A \sin B = 1;$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B = \frac{24}{25}.$$

$$6. \sec A = \frac{17}{8}; \quad \therefore \cos A = \frac{8}{17}, \quad \sin A = \frac{15}{17}.$$

$$\operatorname{cosec} B = \frac{5}{4}; \quad \therefore \sin B = \frac{4}{5}, \quad \cos B = \frac{3}{5}.$$

$$\therefore \sec(A+B) = \frac{1}{\cos A \cos B - \sin A \sin B} = -\frac{85}{36}.$$

$$7. \sin 75^\circ = \sin(90^\circ - 15^\circ) = \cos 15^\circ = \cos(45^\circ - 30^\circ) \\ = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$8. \sin 15^\circ = \cos(90^\circ - 15^\circ) = \cos 75^\circ = \cos(45^\circ + 30^\circ) \\ = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

$$9. \frac{\sin(\alpha+\beta)}{\cos \alpha \cos \beta} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta} = \tan \alpha + \tan \beta.$$

$$10. \frac{\sin(\alpha-\beta)}{\sin \alpha \sin \beta} = \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\sin \alpha \sin \beta} = \cot \beta - \cot \alpha.$$

$$11. \frac{\cos(\alpha-\beta)}{\cos \alpha \sin \beta} = \frac{\cos \alpha \cos \beta + \sin \alpha \sin \beta}{\cos \alpha \sin \beta} = \cot \beta + \tan \alpha.$$

$$12. \cos(A+B) \cos(A-B) \\ = (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B) \\ = \cos^2 A \cos^2 B - \sin^2 A \sin^2 B \\ = \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B \\ = \cos^2 A - \sin^2 B.$$

$$13. \sin(A+B) \sin(A-B) \\ = (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B) \\ = \sin^2 A \cos^2 B - \cos^2 A \sin^2 B \\ = (1 - \cos^2 A) \cos^2 B - \cos^2 A (1 - \cos^2 B) \\ = \cos^2 B - \cos^2 A.$$

$$14. \cos(45^\circ - A) - \sin(45^\circ + A) = \frac{1}{\sqrt{2}} \{\cos A + \sin A - \sin A - \cos A\} = 0.$$

$$15. \cos(45^\circ + A) + \sin(A - 45^\circ) = \frac{1}{\sqrt{2}} (\cos A - \sin A + \sin A - \cos A) = 0.$$

$$16. \text{First side} = \cos A \cos B + \sin A \sin B - \sin A \cos B - \cos A \sin B \\ = (\cos A - \sin A) \cos B - (\cos A - \sin A) \sin B \\ = (\cos A - \sin A) (\cos B - \sin B).$$

$$\begin{aligned} 17. \quad \text{First side} &= \cos A \cos B - \sin A \sin B + \sin A \cos B - \cos A \sin B \\ &= (\cos A + \sin A) \cos B - (\cos A + \sin A) \sin B \\ &= (\cos A + \sin A) (\cos B - \sin B). \end{aligned}$$

$$\begin{aligned} 18. \quad \text{First side} &= 2 (\sin A \cos 45^\circ + \cos A \sin 45^\circ) (\sin A \cos 45^\circ - \cos A \sin 45^\circ) \\ &= 2 \times \frac{1}{\sqrt{2}} (\sin A + \cos A) \times \frac{1}{\sqrt{2}} (\sin A - \cos A) \\ &= \sin^2 A - \cos^2 A. \end{aligned}$$

$$\begin{aligned} 19. \quad \text{First side} &= 2 \cdot \frac{1}{\sqrt{2}} \cdot (\cos \alpha - \sin \alpha) \times \frac{1}{\sqrt{2}} (\cos \alpha + \sin \alpha) \\ &= \cos^2 \alpha - \sin^2 \alpha. \end{aligned}$$

$$\begin{aligned} 20. \quad \text{First side} &= 2 \cdot \frac{1}{\sqrt{2}} \cdot (\cos \alpha + \sin \alpha) \times \frac{1}{\sqrt{2}} (\cos \beta - \sin \beta) \\ &= \cos \alpha \cos \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta - \sin \alpha \sin \beta \\ &= \cos (\alpha + \beta) + \sin (\alpha - \beta). \end{aligned}$$

21. As in Ex. 10, it is easily shewn that the first term of the expression
 $= \tan \beta - \tan \gamma$.

Thus the first side

$$= \tan \beta - \tan \gamma + \tan \gamma - \tan \alpha + \tan \alpha - \tan \beta = 0.$$

EXAMPLES. XI. b. PAGE 100.

$$1. \quad \text{We have } \tan A = \frac{1}{2}, \quad \tan B = \frac{1}{3};$$

$$\therefore \tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1.$$

$$3. \quad \text{We have } \cot A = \frac{5}{7}, \quad \cot B = \frac{7}{5};$$

$$\therefore \cot (A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B} = 0,$$

$$\tan (A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{7}{5} - \frac{5}{7}}{1 + 1} = \frac{12}{35}.$$

$$5. \quad \tan (45^\circ + A) = \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \tan A} = \frac{1 + \tan A}{1 - \tan A}.$$

$$6. \quad \tan(45^\circ - A) = \frac{\tan 45^\circ - \tan A}{1 + \tan 45^\circ \tan A} = \frac{1 - \tan A}{1 + \tan A}.$$

$$7. \quad \cot\left(\frac{\pi}{4} - \theta\right) = \frac{\cot \frac{\pi}{4} \cot \theta + 1}{\cot \theta - \cot \frac{\pi}{4}} = \frac{\cot \theta + 1}{\cot \theta - 1}.$$

$$8. \quad \cot\left(\frac{\pi}{4} + \theta\right) = \frac{\cot \frac{\pi}{4} \cot \theta - 1}{\cot \theta + \cot \frac{\pi}{4}} = \frac{\cot \theta - 1}{\cot \theta + 1}.$$

$$9. \quad \tan 15^\circ = \tan(60^\circ - 45^\circ) = \frac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ} = \frac{\sqrt{3} - 1}{1 + \sqrt{3}} = 2 - \sqrt{3}.$$

$$10. \quad \cot 15^\circ = \cot(45^\circ - 30^\circ) = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3}.$$

$$\begin{aligned} 11. \quad \cos(\overline{A+B+C}) &= \cos A \cos(B+C) - \sin A \sin(B+C) \\ &= \cos A \cos B \cos C - \cos A \sin B \sin C \\ &\quad - \sin A \sin B \cos C - \sin A \cos B \sin C. \\ \sin(\overline{A-B+C}) &= \sin(A-B) \cos C + \cos(A-B) \sin C \\ &= \sin A \cos B \cos C - \cos A \sin B \cos C \\ &\quad + \cos A \cos B \sin C + \sin A \sin B \sin C. \end{aligned}$$

$$\begin{aligned} 12. \quad \tan(\overline{A-B-C}) &= \frac{\tan(A-B) - \tan C}{1 + \tan(A-B) \tan C} \\ &= \frac{\frac{\tan A - \tan B}{1 + \tan A \tan B} - \tan C}{1 + \frac{(\tan A - \tan B) \tan C}{1 + \tan A \tan B}} \\ &= \frac{\tan A - \tan B - \tan C - \tan A \tan B \tan C}{1 + \tan A \tan B - \tan B \tan C + \tan C \tan A}. \end{aligned}$$

$$\begin{aligned} 13. \quad \cot(\overline{A+B+C}) &= \frac{\cot(A+B) \cot C - 1}{\cot C + \cot(A+B)} \\ &= \frac{\frac{(\cot A \cot B - 1) \cot C}{\cot B + \cot A} - 1}{\cot C + \frac{\cot A \cot B - 1}{\cot B + \cot A}} \\ &= \frac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot B \cot C + \cot C \cot A + \cot A \cot B - 1}. \end{aligned}$$

EXAMPLES. XI. c. PAGE 101.

1. First side $= \cos (\overline{A + B} - B) = \cos A.$
2. First side $= \sin (3A - A) = \sin 2A.$
3. First side $= \cos (2a - a) = \cos a.$
4. First side $= \cos (30^\circ + A + \overline{30^\circ - A}) = \cos 60^\circ = \frac{1}{2}.$
5. First side $= \sin (60^\circ - A + \overline{30^\circ + A}) = \sin 90^\circ = 1.$
6. First side $= \cos 2a \cos a - \sin 2a \sin a = \cos (2a + a) = \cos 3a.$
7. First side $= \tan (\overline{a - \beta} + \beta) = \tan a.$
8. First side $= \cot (\overline{a + \beta} - a) = \cot \beta.$
9. First side $= \tan (4A - 3A) = \tan A.$
10.
$$\begin{aligned} \cot \theta - \cot 2\theta &= \frac{\cos \theta}{\sin \theta} - \frac{\cos 2\theta}{\sin 2\theta} = \frac{\sin 2\theta \cos \theta - \cos 2\theta \sin \theta}{\sin \theta \sin 2\theta} \\ &= \frac{\sin (2\theta - \theta)}{\sin \theta \sin 2\theta} = \frac{\sin \theta}{\sin \theta \sin 2\theta} = \operatorname{cosec} 2\theta. \end{aligned}$$
11.
$$\begin{aligned} 1 + \tan 2\theta \tan \theta &= 1 + \frac{\sin 2\theta \sin \theta}{\cos 2\theta \cos \theta} = \frac{\cos \theta \sin 2\theta + \sin \theta \sin 2\theta}{\cos \theta \cos 2\theta} \\ &= \frac{\cos (2\theta - \theta)}{\cos \theta \cos 2\theta} = \sec 2\theta. \end{aligned}$$
12.
$$\begin{aligned} 1 + \cot 2\theta \cot \theta &= \frac{\sin 2\theta \sin \theta + \cos 2\theta \cos \theta}{\sin \theta \sin 2\theta} \\ &= \frac{\cos (2\theta - \theta)}{\sin \theta \sin 2\theta} = \operatorname{cosec} 2\theta \cot \theta. \end{aligned}$$
13. First side $= \sin (2\theta + \theta) = \sin 3\theta$
 $= \sin (4\theta - \theta)$
 $= \sin 4\theta \cos \theta - \cos 4\theta \sin \theta.$
14. First side $= \cos (4a + a) = \cos 5a$
 $= \cos (3a + 2a)$
 $= \cos 3a \cos 2a - \sin 3a \sin 2a.$

EXAMPLES. XI. d. PAGE 104.

1. Here $\cos 2A = 2 \cos^2 A - 1 = -\frac{7}{9}.$

3. We have $\sin A = \frac{3}{5}$, and $\cos A = \frac{4}{5}$;

$$\therefore \sin 2A = 2 \sin A \cos A = \frac{24}{25}.$$

5. By Art. 124, $\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{7}{25}$,

$$\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{24}{25}.$$

7. See Example, Art. 122.

$$8. \frac{\sin 2A}{1 + \cos 2A} = \frac{2 \sin A \cos A}{2 \cos^2 A} = \tan A.$$

$$9. \frac{\sin 2A}{1 - \cos 2A} = \frac{2 \sin A \cos A}{2 \sin^2 A} = \cot A.$$

$$10. \frac{1 - \cos A}{\sin A} = \frac{2 \sin^2 \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \tan \frac{A}{2}.$$

$$11. \frac{1 + \cos A}{\sin A} = \frac{2 \cos^2 \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \cot \frac{A}{2}.$$

$$12. 2 \operatorname{cosec} 2\alpha = \frac{2}{\sin 2\alpha} = \frac{1}{\sin \alpha \cos \alpha} = \sec \alpha \operatorname{cosec} \alpha.$$

$$13. \tan \alpha + \cot \alpha = \frac{\sin^2 \alpha + \cos^2 \alpha}{\sin \alpha \cos \alpha} = \frac{1}{\sin \alpha \cos \alpha} = 2 \operatorname{cosec} 2\alpha.$$

$$14. \cos^4 \alpha - \sin^4 \alpha = (\cos^2 \alpha + \sin^2 \alpha) (\cos^2 \alpha - \sin^2 \alpha) = \cos 2\alpha.$$

$$15. \cot \alpha - \tan \alpha = \frac{\cos^2 \alpha - \sin^2 \alpha}{\sin \alpha \cos \alpha} = \frac{2 \cos 2\alpha}{2 \sin \alpha \cos \alpha} = 2 \cot 2\alpha.$$

$$16. \text{By Art. 116, } \cot 2A = \frac{\cot A \cot A - 1}{\cot A + \cot A} = \frac{\cot^2 A - 1}{2 \cot A}.$$

$$17. \frac{\cot A - \tan A}{\cot A + \tan A} = \frac{\frac{1}{\tan A} - \tan A}{\frac{1}{\tan A} + \tan A} = \frac{1 - \tan^2 A}{1 + \tan^2 A} = \cos 2A.$$

$$18. \frac{1 + \cot^2 A}{2 \cot A} = \frac{\sin^2 A + \cos^2 A}{2 \cot A \sin^2 A} = \frac{1}{2 \sin A \cos A} = \operatorname{cosec} 2A.$$

$$19. \frac{\cot^2 A + 1}{\cot^2 A - 1} = \frac{1 + \tan^2 A}{1 - \tan^2 A} = \sec 2A.$$

$$20. \frac{1 + \sec \theta}{\sec \theta} = \frac{1}{\sec \theta} + 1 = \cos \theta + 1 = 2 \cos^2 \frac{\theta}{2}.$$

$$21. \frac{\sec \theta - 1}{\sec \theta} = 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}.$$

$$22. \frac{2 - \sec^2 \theta}{\sec^2 \theta} = \frac{2}{\sec^2 \theta} - 1 = 2 \cos^2 \theta - 1 = \cos 2\theta.$$

$$23. \frac{\operatorname{cosec}^2 \theta - 2}{\operatorname{cosec}^2 \theta} = 1 - 2 \sin^2 \theta = \cos 2\theta.$$

$$24. \text{First side} = \sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} + 2 \sin \frac{A}{2} \cos \frac{A}{2} = 1 + \sin A.$$

$$25. \text{First side} = \sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} - 2 \sin \frac{A}{2} \cos \frac{A}{2} = 1 - \sin A.$$

$$26. \frac{\cos 2a}{1 + \sin 2a} = \frac{\cos^2 a - \sin^2 a}{(\sin a + \cos a)^2} = \frac{\cos a - \sin a}{\cos a + \sin a} = \frac{1 - \tan a}{1 + \tan a} = \tan (45^\circ - a).$$

$$27. \frac{\cos 2a}{1 - \sin 2a} = \frac{\cos^2 a - \sin^2 a}{(\cos a - \sin a)^2} = \frac{\cos a + \sin a}{\cos a - \sin a} = \cot (45^\circ - a).$$

$$28. \sin 8A = 2 \sin 4A \cos 4A = 4 \sin 2A \cos 2A \cos 4A \\ = 8 \sin A \cos A \cos 2A \cos 4A.$$

$$29. \cos 4A = 2 \cos^2 2A - 1 \\ = 2 (2 \cos^2 A - 1)^2 - 1 \\ = 8 \cos^4 A - 8 \cos^2 A + 1.$$

$$30. \text{Second side} = \cos 2 \left(45^\circ - \frac{A}{2} \right) = \cos (90^\circ - A) = \sin A.$$

$$31. \cos^2 \left(\frac{\pi}{4} - a \right) - \sin^2 \left(\frac{\pi}{4} - a \right) = \cos 2 \left(\frac{\pi}{4} - a \right) = \cos \left(\frac{\pi}{2} - 2a \right) = \sin 2a.$$

$$32. \text{First side} = \frac{1 + \tan A}{1 - \tan A} - \frac{1 - \tan A}{1 + \tan A} = \frac{4 \tan A}{1 - \tan^2 A} = 2 \tan 2A.$$

$$33. \text{First side} = \frac{1 + \tan A}{1 - \tan A} + \frac{1 - \tan A}{1 + \tan A} = \frac{2(1 + \tan^2 A)}{1 - \tan^2 A} \\ = 2 \sec 2A. \quad [\text{Art. 124.}]$$

EXAMPLES. XI. e. PAGE 106.

4.
$$\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = \frac{3 \sin A - 4 \sin^3 A}{\sin A} - \frac{4 \cos^3 A - 3 \cos A}{\cos A}$$

$$= 3 - 4 \sin^2 A - 4 \cos^2 A + 3$$

$$= 6 - 4 (\sin^2 A + \cos^2 A) = 2.$$
5.
$$\cot 3A = \cot (2A + A) = \frac{\cot 2A \cdot \cot A - 1}{\cot 2A + \cot A}$$

$$= \frac{\frac{\cot^2 A - 1}{2 \cot A} \cdot \cot A - 1}{\frac{\cot^2 A - 1}{2 \cot A} + \cot A} = \frac{\cot^3 A - 3 \cot A}{3 \cot^2 A - 1}.$$
6. First side $= \frac{3 \cos a + 4 \cos^3 a - 3 \cos a}{3 \sin a - 3 \sin a + 4 \sin^3 a} = \cot^3 a.$
7. First side $= \frac{3 \sin a - 3 \sin^3 a}{3 \cos a - 3 \cos^3 a} = \frac{\sin a (1 - \sin^2 a)}{\cos a (1 - \cos^2 a)} = \cot a.$
8. First side $= \frac{3 \cos a - 3 \cos^3 a}{\cos a} + \frac{3 \sin a - 3 \sin^3 a}{\sin a}$

$$= 3 - 3 \cos^2 a + 3 - 3 \sin^2 a$$

$$= 6 - 3 (\cos^2 a + \sin^2 a) = 3.$$
9. $\sin 18^\circ + \sin 30^\circ = \frac{\sqrt{5}-1}{4} + \frac{1}{2} = \frac{\sqrt{5}+1}{4} = \sin 54^\circ.$
10. $\cos 36^\circ - \sin 18^\circ = \frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4} = \frac{1}{2}.$
11. $\cos^2 36^\circ + \sin^2 18^\circ = \left(\frac{\sqrt{5}+1}{4}\right)^2 + \left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{3+\sqrt{5}}{8} + \frac{3-\sqrt{5}}{8} = \frac{3}{4}.$
12. $4 \sin 18^\circ \cos 36^\circ = 4 \cdot \frac{\sqrt{5}-1}{4} \cdot \frac{\sqrt{5}+1}{4} = \frac{5-1}{4} = 1.$

EXAMPLES. XI. f. PAGE 108.

1. First side $= \frac{\sin 2A}{\cos 2A} - \frac{\sin A}{\cos A} = \frac{\sin 2A \cos A - \cos 2A \sin A}{\cos A \cos 2A}$

$$= \frac{\sin (2A - A)}{\cos A \cos 2A} = \tan A \sec 2A.$$
2. First side $= \frac{\sin 2A}{\cos 2A} + \frac{\cos A}{\sin A} = \frac{\sin 2A \sin A + \cos 2A \cos A}{\cos 2A \sin A}$

$$= \frac{\cos A}{\cos 2A \sin A} = \cot A \sec 2A.$$

$$3. \text{ First side} = \frac{2 \sin^2 \theta + 2 \sin \theta \cos \theta}{2 \cos^2 \theta + 2 \sin \theta \cos \theta} = \frac{2 \sin \theta (\sin \theta + \cos \theta)}{2 \cos \theta (\sin \theta + \cos \theta)} = \tan \theta.$$

$$4. \text{ First side} = \frac{2 \cos^2 \frac{\theta}{2} + \cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} + \sin \frac{\theta}{2}} = \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = \cot \frac{\theta}{2}.$$

$$\begin{aligned} 5. \cos^6 a - \sin^6 a &= (\cos^2 a - \sin^2 a)(\cos^4 a + \cos^2 a \sin^2 a + \sin^4 a) \\ &= \cos 2a [(\cos^2 a + \sin^2 a)^2 - \sin^2 a \cos^2 a] \\ &= \cos 2a \left(1 - \frac{1}{4} \sin^2 2a\right). \end{aligned}$$

$$\begin{aligned} 6. 4(\cos^6 \theta + \sin^6 \theta) &= 4(\cos^2 \theta + \sin^2 \theta)(\cos^4 \theta + \sin^4 \theta - \sin^2 \theta \cos^2 \theta) \\ &= 4\{(\cos^2 \theta + \sin^2 \theta)^2 - 3 \sin^2 \theta \cos^2 \theta\} \\ &= 4 - 3 \sin^2 2\theta = 1 + 3(1 - \sin^2 2\theta) \\ &= 1 + 3 \cos^2 2\theta. \end{aligned}$$

$$\begin{aligned} 7. \text{ First side} &= \frac{4 \cos^3 a - 3 \cos a + 3 \sin a - 4 \sin^3 a}{\cos a - \sin a} \\ &= \frac{4(\cos^3 a - \sin^3 a) - 3(\cos a - \sin a)}{\cos a - \sin a} \\ &= 4(\cos^2 a + \sin a \cos a + \sin^2 a) - 3 \\ &= 4 + 2 \sin 2a - 3 = 1 + 2 \sin 2a. \end{aligned}$$

$$\begin{aligned} 8. \text{ First side} &= \frac{4 \cos^3 a - 3 \cos a - 3 \sin a + 4 \sin^3 a}{\cos a + \sin a} \\ &= 4(\cos^2 a - \cos a \sin a + \sin^2 a) - 3 \\ &= 4 - 2 \sin 2a - 3 = 1 - 2 \sin 2a. \end{aligned}$$

$$9. \frac{\cos a + \sin a}{\cos a - \sin a} = \frac{(\cos a + \sin a)^2}{\cos^2 a - \sin^2 a} = \frac{\sin 2a + 1}{\cos 2a} = \tan 2a + \sec 2a.$$

$$10. \frac{\cot a - 1}{\cot a + 1} = \frac{\cos a - \sin a}{\cos a + \sin a} = \frac{1 - \sin 2a}{\cos 2a}. \quad [\text{See Example 2, p. 107.}]$$

$$11. \frac{1 + \sin \theta}{\cos \theta} = \frac{\left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2}\right)^2}{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}} = \frac{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}} = \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}}.$$

$$12. \frac{\cos \theta}{1 - \sin \theta} = \frac{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}{\left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2}\right)^2} = \frac{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}} = \frac{\cot \frac{\theta}{2} + 1}{\cot \frac{\theta}{2} - 1}.$$

$$13. \sec A - \tan A = \frac{1 - \sin A}{\cos A} = \frac{1 - \tan \frac{A}{2}}{1 + \tan \frac{A}{2}} = \tan \left(45^\circ - \frac{A}{2} \right).$$

$$14. \tan A + \sec A = \frac{\sin A + 1}{\cos A} = \frac{1 + \tan \frac{A}{2}}{1 - \tan \frac{A}{2}} = \cot \left(45^\circ - \frac{A}{2} \right).$$

$$15. \text{Second side} = \frac{2 \sin^2 \left(\frac{\pi}{4} + \frac{\theta}{2} \right)}{2 \cos^2 \left(\frac{\pi}{4} + \frac{\theta}{2} \right)} = \frac{1 - \cos \left(\frac{\pi}{2} + \theta \right)}{1 + \cos \left(\frac{\pi}{2} + \theta \right)} = \frac{1 + \sin \theta}{1 - \sin \theta}.$$

$$16. (2 \cos A + 1)(2 \cos A - 1) = 4 \cos^2 A - 1 = 2 \cos 2A + 1.$$

$$17. \text{First side} = \frac{2 \sin A \cos A}{2 \cos^2 A} \times \frac{\cos A}{2 \cos^2 \frac{A}{2}} = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \cos^2 \frac{A}{2}} = \tan \frac{A}{2}.$$

$$18. \text{First side} = \frac{2 \sin A \cos A}{2 \sin^2 A} \times \frac{2 \sin^2 \frac{A}{2}}{\cos A} = \frac{2 \sin^2 \frac{A}{2}}{\sin A} = \tan \frac{A}{2}.$$

$$19. \text{First side} = \cos 3a (3 \sin a - \sin 3a) + \sin 3a (3 \cos a + \cos 3a) \\ = 3 (\cos 3a \sin a + \sin 3a \cos a) = 3 \sin 4a.$$

$$20. \text{First side} = \frac{1}{4} (3 \cos a + \cos 3a) \cos 3a + \frac{1}{4} (3 \sin a - \sin 3a) \sin 3a \\ = \frac{3}{4} (\cos a \cos 3a + \sin a \sin 3a) + \frac{1}{4} (\cos^2 3a - \sin^2 3a) \\ = \frac{1}{4} \{ 3 \cos (3a - a) + \cos 6a \} = \frac{1}{4} (3 \cos 2a + \cos 6a) = \cos^3 2a.$$

$$21. \text{First side} = 3 \cos 20^\circ + \cos 60^\circ + 3 \cos 40^\circ + \cos 120^\circ \\ = 3 (\cos 20^\circ + \cos 40^\circ).$$

$$22. \text{First side} = 3 \cos 10^\circ + \cos 30^\circ + 3 \sin 20^\circ - \sin 60^\circ \\ = 3 (\cos 10^\circ + \sin 20^\circ).$$

$$23. \tan 3A = \tan (2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}.$$

Multiply up and transpose; then we obtain

$$\tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A$$

$$\begin{aligned}
 24. \quad \frac{\cot \theta}{\cot \theta - \cot 3\theta} + \frac{\tan \theta}{\tan \theta - \tan 3\theta} &= \frac{\frac{1}{\tan \theta}}{\frac{1}{\tan \theta} - \frac{1}{\tan 3\theta}} + \frac{\tan \theta}{\tan \theta - \tan 3\theta} \\
 &= \frac{\tan 3\theta}{\tan 3\theta - \tan \theta} + \frac{\tan \theta}{\tan \theta - \tan 3\theta} = \frac{\tan 3\theta - \tan \theta}{\tan 3\theta - \tan \theta} = 1. \\
 25. \quad \frac{1}{\tan 3\theta + \tan \theta} - \frac{1}{\cot 3\theta + \cot \theta} &= \frac{1}{\tan 3\theta + \tan \theta} - \frac{\tan 3\theta \tan \theta}{\tan 3\theta + \tan \theta} \\
 &= \frac{1 - \tan 3\theta \tan \theta}{\tan 3\theta + \tan \theta} = \frac{1}{\tan 4\theta} = \cot 4\theta.
 \end{aligned}$$

EXAMPLES. XII. a. PAGE 112.

For Examples 1—16, see Examples on page 111.

$$\begin{aligned}
 17. \quad 2 \cos 2\beta \cos (\alpha - \beta) &= \cos (2\beta + \alpha - \beta) + \cos (2\beta - \alpha + \beta) \\
 &= \cos (\beta + \alpha) + \cos (3\beta - \alpha). \\
 18. \quad 2 \sin 3\alpha \sin (\alpha + \beta) &= \cos (3\alpha - \alpha - \beta) - \cos (3\alpha + \alpha + \beta) \\
 &= \cos (2\alpha - \beta) - \cos (4\alpha + \beta). \\
 19. \quad 2 \sin (2\theta + \phi) \cos (\theta - 2\phi) &= \sin (2\theta + \phi + \theta - 2\phi) + \sin (2\theta + \phi - \theta + 2\phi) \\
 &= \sin (3\theta - \phi) + \sin (\theta + 3\phi). \\
 20. \quad 2 \cos (3\theta + \phi) \sin (\theta - 2\phi) &= \sin (3\theta + \phi + \theta - 2\phi) - \sin (3\theta + \phi - \theta + 2\phi) \\
 &= \sin (4\theta - \phi) - \sin (2\theta + 3\phi). \\
 21. \quad \cos (60^\circ + \alpha) \sin (60^\circ - \alpha) &= \frac{1}{2} \{ \sin 120^\circ - \sin 2\alpha \} = \frac{1}{2} \left(\frac{\sqrt{3}}{2} - \sin 2\alpha \right).
 \end{aligned}$$

EXAMPLES. XII. b. PAGE 114.

For Examples 1—12, see Examples on page 113.

$$\begin{aligned}
 13. \quad \frac{\cos a - \cos 3a}{\sin 3a - \sin a} &= \frac{2 \sin 2a \sin a}{2 \cos 2a \sin a} = \tan 2a. \\
 14. \quad \frac{\sin 2a + \sin 3a}{\cos 2a - \cos 3a} &= \frac{2 \sin \frac{5a}{2} \cos \frac{a}{2}}{2 \sin \frac{5a}{2} \sin \frac{a}{2}} = \cot \frac{a}{2}. \\
 15. \quad \frac{\cos 4\theta - \cos \theta}{\sin \theta - \sin 4\theta} &= \frac{-2 \sin \frac{5\theta}{2} \sin \frac{3\theta}{2}}{-2 \cos \frac{5\theta}{2} \sin \frac{3\theta}{2}} = \tan \frac{5\theta}{2}.
 \end{aligned}$$

16. $\frac{\cos 2\theta - \cos 12\theta}{\sin 12\theta + \sin 2\theta} = \frac{2 \sin 7\theta \sin 5\theta}{2 \sin 7\theta \cos 5\theta} = \tan 5\theta.$
17. First side $= 2 \cos 60^\circ \sin A = 2 \times \frac{1}{2} \sin A = \sin A.$
18. First side $= 2 \cos 30^\circ \cos A = \sqrt{3} \cos A.$
19. First side $= 2 \sin \frac{\pi}{4} \sin (-a) = -2 \times \frac{1}{\sqrt{2}} \sin a = -\sqrt{2} \sin a.$
20. First side $= \frac{2 \cos a \cos (a - 3\beta)}{2 \sin a \cos (a - 3\beta)} = \cot a.$
21. First side $= \frac{2 \sin (2\theta - \phi) \sin (\theta + 2\phi)}{2 \sin (2\theta - \phi) \cos (\theta + 2\phi)} = \tan (\theta + 2\phi).$
22. First side $= \frac{2 \cos \frac{a+5\beta}{2} \sin \frac{a-3\beta}{2}}{2 \cos \frac{a+5\beta}{2} \cos \frac{a-3\beta}{2}} = \tan \frac{a-3\beta}{2}.$

EXAMPLES. XII. c. PAGE 116.

1. $\cos 3A + \sin 2A - \sin 4A = \cos 3A - 2 \cos 3A \sin A$
 $= \cos 3A (1 - 2 \sin A).$
2. $\sin 3\theta - \sin \theta - \sin 5\theta = \sin 3\theta - 2 \sin 3\theta \cos 2\theta$
 $= \sin 3\theta (1 - 2 \cos 2\theta).$
3. $\cos \theta + \cos 2\theta + \cos 5\theta = \cos 2\theta + 2 \cos 3\theta \cos 2\theta$
 $= \cos 2\theta (1 + 2 \cos 3\theta).$
4. $\sin a - \sin 2a + \sin 3a = \sin a + 2 \cos \frac{5a}{2} \sin \frac{a}{2}$
 $= 2 \sin \frac{a}{2} \left(\cos \frac{a}{2} + \cos \frac{5a}{2} \right)$
 $= 4 \sin \frac{a}{2} \cos a \cos \frac{3a}{2}.$
5. $\sin 3a + \sin 7a + \sin 10a = 2 \sin 5a \cos 2a + \sin 10a$
 $= 2 \sin 5a (\cos 2a + \cos 5a)$
 $= 2 \sin 5a \cos \frac{7a}{2} \cos \frac{3a}{2}.$
6. $\sin A + 2 \sin 3A + \sin 5A = 2 \sin 3A + 2 \sin 3A \cos 2A$
 $= 2 \sin 3A (1 + \cos 2A)$
 $= 4 \sin 3A \cos^2 A.$

$$7. \text{ First side} = \frac{\sin 2a + 2 \cos 3a \sin 2a}{\cos 2a + 2 \cos 3a \cos 2a} = \frac{\sin 2a (1 + 2 \cos 3a)}{\cos 2a (1 + 2 \cos 3a)} = \tan 2a.$$

$$8. \text{ First side} = \frac{(\sin a + \sin 5a) + (\sin 2a + \sin 4a)}{(\cos a + \cos 5a) + (\cos 2a + \cos 4a)} \\ = \frac{2 \sin 3a \cos 2a + 2 \sin 3a \cos a}{2 \cos 3a \cos 2a + 2 \cos 3a \cos a} = \frac{\sin 3a}{\cos 3a} = \tan 3a.$$

$$9. \text{ First side} = \frac{2 \cos 5\theta \cos 2\theta - 2 \cos 3\theta \cos 2\theta}{2 \cos 5\theta \sin 2\theta - 2 \cos 3\theta \sin 2\theta} = \cot 2\theta.$$

$$10. \text{ First side} = \frac{1}{2} (\sin 5A - \sin A) - \frac{1}{2} (\sin 5A - \sin 3A) \\ = \frac{1}{2} (\sin 3A - \sin A) = \cos 2A \sin A.$$

$$11. \text{ First side} = \frac{1}{2} (\cos 7A + \cos 3A) - \frac{1}{2} (\cos 7A + \cos A) \\ = \frac{1}{2} (\cos 3A - \cos A) = -\sin 2A \sin A.$$

$$12. \text{ First side} = \frac{1}{2} (\sin 5\theta + \sin 3\theta) - \frac{1}{2} (\sin 5\theta + \sin \theta) \\ = \frac{1}{2} (\sin 3\theta - \sin \theta) = \cos 2\theta \sin \theta.$$

$$13. \cos 5^\circ - \sin 25^\circ = \cos 5^\circ - \cos 65^\circ \\ = 2 \sin 35^\circ \sin 30^\circ = \sin 35^\circ.$$

$$14. \sin 65^\circ + \cos 65^\circ = \sin 65^\circ + \sin 25^\circ \\ = 2 \sin 45^\circ \cos 20^\circ = \sqrt{2} \cos 20^\circ.$$

$$15. \text{ First side} = 2 \cos 60^\circ \cos 20^\circ - \cos 20^\circ \\ = \cos 20^\circ - \cos 20^\circ = 0.$$

$$16. \text{ First side} = 2 \cos 48^\circ \sin 30^\circ + \cos (180^\circ - 48^\circ) \\ = \cos 48^\circ - \cos 48^\circ = 0.$$

$$17. \sin^2 5A - \sin^2 2A = \sin (5A + 2A) \sin (5A - 2A) = \sin 7A \sin 3A.$$

$$18. \cos 2A \cos 5A = \frac{1}{2} (\cos 7A + \cos 3A) = \frac{1}{2} \left(2 \cos^2 \frac{7A}{2} - 1 + 1 - 2 \sin^2 \frac{3A}{2} \right) \\ = \cos^2 \frac{7A}{2} - \sin^2 \frac{3A}{2}.$$

$$19. \text{ First side} = 2 \sin a \cos (\beta + \gamma) + 2 \sin a \cos (\beta - \gamma) \\ = 2 \sin a \{ \cos (\beta + \gamma) + \cos (\beta - \gamma) \} \\ = 4 \sin a \cos \beta \cos \gamma.$$

20. First side $= 2 \sin \gamma \sin (\alpha - \beta) + 2 \sin (\alpha + \beta) \sin \gamma$
 $= 2 \sin \gamma \{ \sin (\alpha - \beta) + \sin (\alpha + \beta) \}$
 $= 4 \sin \alpha \cos \beta \sin \gamma.$
21. First side $= 2 \sin (\alpha + \beta) \cos (\alpha - \beta) - 2 \sin (\alpha + \beta) \cos (\alpha + \beta + 2\gamma)$
 $= 2 \sin (\alpha + \beta) \{ \cos (\alpha - \beta) - \cos (\alpha + \beta + 2\gamma) \}$
 $= 4 \sin (\alpha + \beta) \sin (\beta + \gamma) \sin (\gamma + \alpha).$
22. First side $= 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} + 2 \cos \frac{\alpha + \beta + 2\gamma}{2} \cos \frac{\alpha + \beta}{2}$
 $= 2 \cos \frac{\alpha + \beta}{2} \left(\cos \frac{\alpha - \beta}{2} + \cos \frac{\alpha + \beta + 2\gamma}{2} \right)$
 $= 4 \cos \frac{\beta + \gamma}{2} \cos \frac{\gamma + \alpha}{2} \cos \frac{\alpha + \beta}{2}.$
23. First side $= 2 \sin A \{ \cos 2A - \cos 120^\circ \}$
 $= 2 \sin A \left\{ \cos 2A + \frac{1}{2} \right\}$
 $= 2 \sin A \cos 2A + \sin A$
 $= \sin 3A - \sin A + \sin A = \sin 3A.$
24. First side $= 2 \cos \theta \left\{ \cos \frac{4\pi}{3} + \cos 2\theta \right\} = 2 \cos \theta \left(-\frac{1}{2} + \cos 2\theta \right)$
 $= -\cos \theta + 2 \cos \theta \cos 2\theta$
 $= -\cos \theta + \cos 3\theta + \cos \theta$
 $= \cos 3\theta.$
25. First side $= \cos \theta + 2 \cos \frac{2\pi}{3} \cos \theta$
 $= \cos \theta - \cos \theta = 0.$
26. First side $= \frac{1}{2} \{ 1 + \cos 2A + 1 + \cos 2(60^\circ + A) + 1 + \cos 2(60^\circ - A) \}$
 $= \frac{1}{2} \{ 3 + \cos 2A + 2 \cos 120^\circ \cos 2A \}$
 $= \frac{1}{2} \{ 3 + \cos 2A - \cos 2A \} = \frac{3}{2}.$
27. First side $= \frac{1}{2} \{ 3 - \cos 2A - \cos 2(120^\circ + A) - \cos 2(120^\circ - A) \}$
 $= \frac{1}{2} \{ 3 - \cos 2A - 2 \cos 240^\circ \cos 2A \}$
 $= \frac{1}{2} \{ 3 - \cos 2A + \cos 2A \} = \frac{3}{2}.$

$$\begin{aligned}
 28. \quad \cos 20^\circ \cos 40^\circ \cos 80^\circ &= \frac{1}{2} \cos 20^\circ (\cos 120^\circ + \cos 40^\circ) \\
 &= \frac{1}{2} \cos 20^\circ \left(-\frac{3}{2} + 2 \cos^2 20^\circ \right) \\
 &= \frac{1}{4} (4 \cos^3 20^\circ - 3 \cos 20^\circ) = \frac{1}{4} \cos 60^\circ = \frac{1}{8}.
 \end{aligned}$$

$$\begin{aligned}
 29. \quad \sin 20^\circ \sin 40^\circ \sin 80^\circ &= \frac{1}{2} \sin 20^\circ \{ \cos 40^\circ - \cos 120^\circ \} \\
 &= \frac{1}{2} \sin 20^\circ \left\{ \frac{3}{2} - 2 \sin^2 20^\circ \right\} \\
 &= \frac{1}{4} \sin 60^\circ = \frac{\sqrt{3}}{8}.
 \end{aligned}$$

EXAMPLES. XII. d. PAGE 119.

$$\begin{aligned}
 1. \quad \text{First side} &= 2 \cos (A+B) \sin (A-B) + 2 \sin C \cos C \\
 &= -2 \cos C \sin (A-B) + 2 \sin (A+B) \cos C \\
 &= 2 \cos C \{ \sin (A+B) - \sin (A-B) \} \\
 &= 4 \cos A \sin B \cos C.
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \text{First side} &= 2 \cos (A+B) \sin (A-B) - 2 \sin C \cos C \\
 &= -2 \cos C \{ \sin (A-B) + \sin (A+B) \} \\
 &= -4 \sin A \cos B \cos C.
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \text{First side} &= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2} \\
 &= 2 \cos \frac{C}{2} \left\{ \cos \frac{A-B}{2} + \cos \frac{A+B}{2} \right\} \\
 &= 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \text{First side} &= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} - 2 \sin \frac{C}{2} \cos \frac{C}{2} \\
 &= 2 \cos \frac{C}{2} \left\{ \cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right\} \\
 &= 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \text{First side} &= 2 \sin \frac{A+B}{2} \sin \frac{B-A}{2} + 2 \cos^2 \frac{C}{2} - 1 \\
 &= 2 \cos \frac{C}{2} \left\{ \sin \frac{B-A}{2} + \sin \frac{B+A}{2} \right\} - 1 \\
 &= 4 \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} - 1.
 \end{aligned}$$

$$6. \quad \text{First side} = \frac{4 \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \tan \frac{B}{2} \tan \frac{C}{2}.$$

$$7. \quad \text{We have} \quad \tan \frac{A+B}{2} \tan \frac{C}{2} = 1;$$

$$\therefore \frac{\left(\tan \frac{A}{2} + \tan \frac{B}{2} \right) \tan \frac{C}{2}}{1 - \tan \frac{A}{2} \tan \frac{B}{2}} = 1.$$

Multiply up and transpose and we obtain the required result.

$$\begin{aligned} 8. \quad \text{First side} &= \frac{2 \cos^2 \frac{A}{2} + 2 \sin \frac{B+C}{2} \sin \frac{B-C}{2}}{2 \cos^2 \frac{A}{2} - 2 \sin \frac{B+C}{2} \sin \frac{B-C}{2}} \\ &= \frac{\sin \frac{B+C}{2} + \sin \frac{B-C}{2}}{\sin \frac{B+C}{2} - \sin \frac{B-C}{2}} = \frac{2 \sin \frac{B}{2} \cos \frac{C}{2}}{2 \cos \frac{B}{2} \sin \frac{C}{2}} = \tan \frac{B}{2} \cot \frac{C}{2}. \end{aligned}$$

$$\begin{aligned} 9. \quad \text{First side} &= 2 \cos (A+B) \cos (A-B) + 2 \cos^2 C + 4 \cos A \cos B \cos C \\ &= -2 \cos C \{ \cos (A-B) + \cos (A+B) \} + 4 \cos A \cos B \cos C \\ &= -4 \cos A \cos B \cos C + 4 \cos A \cos B \cos C = 0. \end{aligned}$$

$$10. \quad \text{We have} \quad \cot (A+B) = \cot (180^\circ - C) = -\cot C;$$

$$\therefore \frac{\cot A \cot B - 1}{\cot A + \cot B} = -\cot C;$$

whence by multiplying up and transposing we obtain the required result.

$$\begin{aligned} 11. \quad \text{First side} &= \frac{\sin (B+C)}{\sin B \sin C} \cdot \frac{\sin (C+A)}{\sin C \sin A} \cdot \frac{\sin (A+B)}{\sin A \sin B} \\ &= \frac{\sin A \sin B \sin C}{\sin^2 A \sin^2 B \sin^2 C} = \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C. \end{aligned}$$

$$\begin{aligned} 12. \quad \text{First side} &= \frac{1}{2} \{ 3 + \cos 2A + \cos 2B + \cos 2C + 4 \cos A \cos B \cos C \} \\ &= \frac{1}{2} (3 - 1), \text{ by Example 9,} \\ &= 1. \end{aligned}$$

$$\begin{aligned}
 13. \quad \text{First side} &= \frac{1}{2}(3 - \cos A - \cos B - \cos C) \\
 &= \frac{1}{2}\left(3 - 1 - 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}\right) \quad [\text{See Ex. 3, p. 119.}] \\
 &= 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \text{First side} &= \frac{1}{2}\{2 + 2 \cos^2 2A + \cos 4B + \cos 4C\} \\
 &= \frac{1}{2}\{2 + 2 \cos^2 2A + 2 \cos 2(B+C) \cos 2(B-C)\} \\
 &= 1 + \cos 2A \{\cos 2(B+C) + \cos 2(B-C)\} \\
 &= 1 + 2 \cos 2A \cos 2B \cos 2C.
 \end{aligned}$$

$$\begin{aligned}
 15. \quad \text{First side} &= \frac{\cot B + \cot C}{\frac{1}{\cot B} + \frac{1}{\cot C}} + \frac{\cot C + \cot A}{\frac{1}{\cot C} + \frac{1}{\cot A}} + \frac{\cot A + \cot B}{\frac{1}{\cot A} + \frac{1}{\cot B}} \\
 &= \cot B \cot C + \cot C \cot A + \cot A \cot B \\
 &= 1. \quad [\text{See Ex. 10.}]
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \text{First side} &= \frac{\tan A \tan B \tan C}{16 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} \\
 &= \frac{\sin A \sin B \sin C}{\cos A \cos B \cos C} \cdot \frac{1}{16 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} \\
 &= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2} \cdot 2 \sin \frac{B}{2} \cos \frac{B}{2} \cdot 2 \sin \frac{C}{2} \cos \frac{C}{2}}{\cos A \cos B \cos C \cdot 16 \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}} \\
 &= \frac{\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}}{2 \cos A \cos B \cos C}.
 \end{aligned}$$

EXAMPLES. XII. e. PAGE 121.

$$\begin{aligned}
 1. \quad \text{First side} &= \frac{1}{2}\{\sin 2\alpha - \sin 2\beta + \sin 2\beta - \sin 2\gamma + \sin 2\gamma \\
 &\quad - \sin 2\delta + \sin 2\delta - \sin 2\alpha\} = 0.
 \end{aligned}$$

$$2. \quad \text{First side} = \cot \gamma - \cot \beta + \cot \alpha - \cot \gamma + \cot \beta - \cot \alpha = 0.$$

[See Ex. XI. a, 10.]

$$\begin{aligned}
 3. \quad \text{First side} &= \frac{2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} + 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha+\beta}{2}}{2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} - 2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha+\beta}{2}} \\
 &= \frac{\cos \frac{\alpha-\beta}{2} + \cos \frac{\alpha+\beta}{2}}{\cos \frac{\alpha-\beta}{2} - \cos \frac{\alpha+\beta}{2}} = \cot \frac{\alpha}{2} \cot \frac{\beta}{2}.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \text{First side} &= \sin \alpha (\cos \beta \cos \gamma - \sin \beta \sin \gamma) - \sin \beta (\cos \alpha \cos \gamma - \sin \alpha \sin \gamma) \\
 &= \cos \gamma (\sin \alpha \cos \beta - \cos \alpha \sin \beta) = \cos \gamma \sin (\alpha - \beta).
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \text{First side} &= \cos \alpha (\cos \beta \cos \gamma - \sin \beta \sin \gamma) - \cos \beta (\cos \alpha \cos \gamma - \sin \alpha \sin \gamma) \\
 &= \sin \gamma (\sin \alpha \cos \beta - \cos \alpha \sin \beta) = \sin \gamma \sin (\alpha - \beta).
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \text{First side} &= \cos A \cos 2A + \sin A \sin 2A - (\sin A \cos 2A + \cos A \sin 2A) \\
 &= \cos (2A - A) - \sin (2A + A) \\
 &= \cos A - \sin 3A.
 \end{aligned}$$

$$\begin{aligned}
 7. \quad a \cos 2\theta + b \sin 2\theta &= \frac{a(1 - \tan^2 \theta)}{1 + \tan^2 \theta} + \frac{2b \tan \theta}{1 + \tan^2 \theta}, & [\text{Art. 124.}] \\
 &= \frac{a(a^2 - b^2)}{a^2 + b^2} + \frac{2ab^2}{a^2 + b^2} = a \cdot \frac{a^2 + b^2}{a^2 + b^2} = a.
 \end{aligned}$$

$$8. \quad \sin 2A + \cos 2A = \frac{2 \tan A}{1 + \tan^2 A} + \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{(1 + \tan A)^2 - 2 \tan^2 A}{1 + \tan^2 A}.$$

$$9. \quad \sin 4A = 2 \sin 2A \cos 2A = \frac{4 \tan A (1 - \tan^2 A)}{(1 + \tan^2 A)^2}.$$

$$10. \quad \text{We have } A + B = 45^\circ;$$

$$\therefore \tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1;$$

$$\therefore \tan A + \tan B + \tan A \tan B = 1.$$

$$\therefore (1 + \tan A)(1 + \tan B) = 1 + \tan A + \tan B + \tan A \tan B = 2.$$

$$\begin{aligned}
 11. \quad \text{First side} &= \frac{\cos (15^\circ - A) \cos (15^\circ + A) + \sin (15^\circ - A) \sin (15^\circ + A)}{\sin (15^\circ - A) \cos (15^\circ + A)} \\
 &= \frac{2 \cos 2A}{\sin 30^\circ - \sin 2A} = \frac{4 \cos 2A}{1 - 2 \sin 2A}.
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \text{First side} &= \frac{\cos^2 (15^\circ + A) + \sin^2 (15^\circ + A)}{\sin (15^\circ + A) \cos (15^\circ + A)} = \frac{2}{\sin (30^\circ + 2A)} \\
 &= \frac{2}{\frac{1}{2} \cos 2A + \frac{\sqrt{3}}{2} \sin 2A} = \frac{4}{\cos 2A + \sqrt{3} \sin 2A}.
 \end{aligned}$$

$$\begin{aligned}
 13. \quad \text{First side} &= \frac{\sin(A + 30^\circ) \sin(A - 30^\circ)}{\cos(A + 30^\circ) \cos(A - 30^\circ)} \\
 &= \frac{\cos 60^\circ - \cos 2A}{\cos 60^\circ + \cos 2A} = \frac{1 - 2 \cos 2A}{1 + 2 \cos 2A}.
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \text{First side} &= (4 \cos^2 A - 1)(2 \cos 2A - 1) \\
 &= (2 \cos 2A + 1)(2 \cos 2A - 1) \\
 &= 4 \cos^2 2A - 1 = 2 \cos 4A + 1.
 \end{aligned}$$

$$\begin{aligned}
 15. \quad \text{We have } \tan(\theta + \phi + \psi) &= \frac{\tan \theta + \tan \phi + \tan \psi - \tan \theta \tan \phi \tan \psi}{1 - \tan \theta \tan \phi - \tan \phi \tan \psi - \tan \psi \tan \theta}, \\
 \therefore \text{ if } \theta + \phi + \psi &= 0,
 \end{aligned}$$

we have $\tan \theta + \tan \phi + \tan \psi = \tan \theta \tan \phi \tan \psi.$

Now putting $\theta = \beta - \gamma, \phi = \gamma - \alpha, \psi = \alpha - \beta,$

we have at once the required identity.

16. First side

$$\begin{aligned}
 &= 2 \sin \frac{\beta - \alpha}{2} \cos \frac{\alpha + \beta - 2\gamma}{2} + \sin(\alpha - \beta) + 4 \sin \frac{\beta - \gamma}{2} \sin \frac{\gamma - \alpha}{2} \sin \frac{\alpha - \beta}{2} \\
 &= -2 \sin \frac{\alpha - \beta}{2} \left\{ \cos \frac{\alpha + \beta - 2\gamma}{2} - \cos \frac{\alpha - \beta}{2} \right\} + 4 \sin \frac{\beta - \gamma}{2} \sin \frac{\gamma - \alpha}{2} \sin \frac{\alpha - \beta}{2} \\
 &= -4 \sin \frac{\alpha - \beta}{2} \sin \frac{\alpha - \gamma}{2} \sin \frac{\gamma - \beta}{2} + 4 \sin \frac{\beta - \gamma}{2} \sin \frac{\gamma - \alpha}{2} \sin \frac{\alpha - \beta}{2} = 0.
 \end{aligned}$$

$$\begin{aligned}
 17. \quad \text{First side} &= \frac{1}{2} \{ \cos 2(\beta - \gamma) + \cos 2(\gamma - \alpha) + 2 \cos^2(\alpha - \beta) + 2 \} \\
 &= \frac{1}{2} \{ 2 \cos(\alpha - \beta) \cos(\beta - 2\gamma + \alpha) + 2 \cos^2(\alpha - \beta) + 2 \} \\
 &= \cos(\alpha - \beta) [\cos(\beta - 2\gamma + \alpha) + \cos(\alpha - \beta)] + 1 \\
 &= 1 + 2 \cos(\beta - \gamma) \cos(\gamma - \alpha) \cos(\alpha - \beta).
 \end{aligned}$$

$$\begin{aligned}
 18. \quad \text{First side} &= 1 + \frac{1}{2} (\cos 2\alpha + \cos 2\beta) - 2 \cos \alpha \cos \beta \cos(\alpha + \beta) \\
 &= 1 + \cos(\alpha + \beta) \cos(\alpha - \beta) - 2 \cos \alpha \cos \beta \cos(\alpha + \beta) \\
 &= 1 + \cos(\alpha + \beta) \{ \sin \alpha \sin \beta - \cos \alpha \cos \beta \} \\
 &= 1 - \cos^2(\alpha + \beta) = \sin^2(\alpha + \beta).
 \end{aligned}$$

$$\begin{aligned}
 19. \quad \text{First side} &= 1 - \frac{1}{2} (\cos 2\alpha + \cos 2\beta) + 2 \sin \alpha \sin \beta \cos(\alpha + \beta) \\
 &= 1 - \cos(\alpha + \beta) \{ \cos(\alpha - \beta) - 2 \sin \alpha \sin \beta \} \\
 &= 1 - \cos^2(\alpha + \beta) = \sin^2(\alpha + \beta).
 \end{aligned}$$

$$\begin{aligned}
 20. \quad \text{First side} &= \cos 12^\circ + 2 \cos 72^\circ \cos 12^\circ \\
 &= \cos 12^\circ \left(1 + \frac{\sqrt{5}-1}{2} \right) = \cos 12^\circ \left(\frac{\sqrt{5}+1}{2} \right) \\
 &= 2 \cos 12^\circ \cos 36^\circ = \cos 24^\circ + \cos 48^\circ.
 \end{aligned}$$

$$\begin{aligned}
 21. \quad \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} &= 4 \cos \frac{\pi-A}{4} \cos \frac{\pi-B}{4} \cos \frac{\pi-C}{4} \quad [\text{Ex. 2, p. 121.}] \\
 &= 4 \cos \frac{B+C}{4} \cos \frac{C+A}{4} \cos \frac{A+B}{4}.
 \end{aligned}$$

$$\begin{aligned}
 22. \quad \text{Second side} &= 2 \cos \frac{\pi-B}{4} \left[\cos \frac{2\pi+A+C}{4} + \cos \frac{A-C}{4} \right] \\
 &= 2 \cos \frac{A+C}{4} \cos \left(\frac{\pi}{2} + \frac{A+C}{4} \right) + 2 \cos \frac{A+C}{4} \cos \frac{A-C}{4} \\
 &= -2 \cos \frac{A+C}{4} \sin \frac{A+C}{4} + \cos \frac{A}{2} + \cos \frac{C}{2} \\
 &= -\sin \frac{A+C}{2} + \cos \frac{A}{2} + \cos \frac{C}{2} \\
 &= \cos \frac{A}{2} - \cos \frac{B}{2} + \cos \frac{C}{2}.
 \end{aligned}$$

23. This may be done in the same way as the two preceding examples. The following solution exhibits another method.

From Example 3 of Art. 135,

$$\text{if } \alpha + \beta + \gamma = \pi, \text{ then } \cos \alpha + \cos \beta + \cos \gamma = 1 + 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}.$$

$$\text{Put } \alpha = \frac{\pi}{2} - \frac{A}{2}, \quad \beta = \frac{\pi}{2} - \frac{B}{2}, \quad \gamma = \frac{\pi}{2} - \frac{C}{2};$$

then $\alpha + \beta + \gamma = \pi$, and after substituting for α, β, γ ,

$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} = 1 + 4 \sin \frac{\pi-A}{4} \sin \frac{\pi-B}{4} \sin \frac{\pi-C}{4}.$$

$$\begin{aligned}
 24. \quad \text{First side} &= \frac{2 \sin (\alpha + \beta) \cos (\alpha - \beta) + \sin 2\gamma}{2 \sin (\alpha + \beta) \cos (\alpha - \beta) + \sin 2\gamma} = \frac{\cos (\alpha - \beta) + \sin \gamma}{\cos (\alpha - \beta) + \sin \gamma} \\
 &= \frac{\cos (\alpha - \beta) + \cos (\alpha + \beta)}{\cos (\alpha - \beta) - \cos (\alpha + \beta)} = \cot \alpha \cot \beta.
 \end{aligned}$$

$$25. \quad \text{Here } \tan (\alpha + \beta) = \tan \left(\frac{\pi}{2} - \gamma \right) = \cot \gamma;$$

$$\therefore \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{1}{\tan \gamma}.$$

Multiply up, and transpose.

EXAMPLES. XII. f. PAGE 122 A.

9. Use the formulæ of Arts. 123, 124.

$$10. \quad \frac{\sin 3A}{\sin 2A - \sin A} = \frac{3 \sin A - 4 \sin^3 A}{2 \sin A \cos A - \sin A}$$

$$= \frac{3 - 4 \sin^2 A}{2 \cos A - 1} = \frac{4 \cos^2 A - 1}{2 \cos A - 1} = 2 \cos A + 1.$$

$$11. \quad \text{Here } \cos \alpha = \sqrt{1 - \frac{7^2}{25^2}} = \frac{24}{25}. \quad \sin \beta = \sqrt{1 - \frac{3^2}{5^2}} = \frac{4}{5}.$$

$$\therefore \cos(\alpha + \beta) = \frac{24}{25} \cdot \frac{6}{10} - \frac{7}{25} \cdot \frac{8}{10} = \frac{88}{250} = \cdot 352;$$

whence, by the Tables, $\alpha + \beta = 69^\circ 23'.$

Again, $\sin \alpha = \cdot 28$ gives $\alpha = 16^\circ 15',$
 $\cos \beta = \cdot 6$ gives $\beta = 53^\circ 8';$
 $\therefore \alpha + \beta = 69^\circ 23'.$

$$12. \quad \sin(\alpha + \beta + \gamma) = \sin 2\alpha = 2 \sin \alpha \cos \alpha = 2 \sin \alpha \cos(\beta + \gamma),$$

$$\sin(\alpha + \beta - \gamma) + \sin(\alpha - \beta + \gamma) = 2 \sin \alpha \cos(\beta - \gamma);$$

$$\therefore, \text{ by addition, first side} = 2 \sin \alpha \{ \cos(\beta + \gamma) + \cos(\beta - \gamma) \}$$

$$= 4 \sin \alpha \cos \beta \cos \gamma.$$

$$13. \quad \cos 57^\circ + \sin 27^\circ = \cos 57^\circ + \cos 63^\circ$$

$$= 2 \cos 60^\circ \cos 3^\circ$$

$$= \cos 3^\circ.$$

Again, $\cos 57^\circ = \cdot 5446,$
 $\cos 63^\circ = \cdot 4540;$
 $\therefore \cos 57^\circ + \cos 63^\circ = \cdot 9986 = \cos 3^\circ, \text{ by the Tables.}$

$$14. \quad \text{Expression} = 2 \sin 5a (\cos 5a + \cos a)$$

$$= \sin 10a + \sin 6a + \sin 4a.$$

$$15. \quad \text{Expression} = 2 \cos 20^\circ (\cos 70^\circ + \cos 10^\circ)$$

$$= \cos 90^\circ + \cos 50^\circ + \cos 30^\circ + \cos 10^\circ$$

$$= \cos 50^\circ + \cos 30^\circ + \cos 10^\circ$$

$$= \cdot 6428 + \cdot 8660 + \cdot 9848$$

$$= 2\cdot 4936.$$

16. See MISCELLANEOUS EXAMPLES K. No. 242.

$$\begin{aligned}
 17. \quad (i) \quad \text{First side} &= x^2 + y^2 + xy (\cot^2 a + \tan^2 a) \\
 &= (x+y)^2 - 2xy + xy \left(\frac{1}{\tan^2 a} + \tan^2 a \right) \\
 &= (x+y)^2 + xy \left(\frac{1 + \tan^4 a}{\tan^2 a} - 2 \right) \\
 &= (x+y)^2 + 4xy \frac{(1 - \tan^2 a)^2}{4 \tan^2 a} \\
 &= (x+y)^2 + 4xy \cot^2 2a.
 \end{aligned}$$

(ii) Use the formula of Ex. 12 on p. 97.

$$\begin{aligned}
 (iii) \quad (\cos a + \cos 7a) + \cos (3a + \cos 5a) \\
 &= 2 \cos 4a \cos 3a + 2 \cos 4a \cos a \\
 &= 2 \cos 4a (\cos 3a + \cos a) \\
 &= \frac{\sin 8a}{\sin 4a} \cdot 2 \cos 2a \cdot \cos a \\
 &= \frac{\sin 8a \cdot 2 \cos 2a \cos a}{2 \sin 2a \cos 2a} \\
 &= \frac{1}{2} \sin 8a \cdot \frac{2 \cos a}{\sin 2a} = \frac{1}{2} \sin 8a \operatorname{cosec} a.
 \end{aligned}$$

$$\begin{aligned}
 18. \quad \sin 2\theta &= 2 \sin \theta \cos \theta = 2 \cdot \frac{6}{10} \cdot \frac{8}{10} = \cdot 96, \\
 \cos 2\theta &= 2 \cos^2 \theta - 1 = 2 \cdot \frac{64}{100} - 1 = \cdot 28.
 \end{aligned}$$

Again, $\cos \theta = \cdot 8$ gives $\theta = 36^\circ 52'$. $\therefore 2\theta = 73^\circ 44'$,
and from the Tables, $\sin 73^\circ 44' = \cdot 96$, $\cos 73^\circ 44' = \cdot 28$.

$$\begin{aligned}
 19. \quad \text{Here} \quad \cos A &= \cos \left(45^\circ - \frac{B}{2} \right) = \frac{1}{\sqrt{2}} \left(\cos \frac{B}{2} + \sin \frac{B}{2} \right) \\
 &= \frac{1}{\sqrt{2}} \sqrt{\cos^2 \frac{B}{2} + \sin^2 \frac{B}{2} + \sin B} = \sqrt{\frac{1 + \sin B}{2}}.
 \end{aligned}$$

$$\begin{aligned}
 20. \quad (i) \quad \text{First side} &= \frac{\cos 10^\circ - \sqrt{3} \sin 10^\circ}{\sin 10^\circ \cos 10^\circ} = \frac{4 \left(\frac{1}{2} \cos 10^\circ - \frac{\sqrt{3}}{2} \sin 10^\circ \right)}{\sin 20^\circ} \\
 &= \frac{4 \sin (30^\circ - 10^\circ)}{\sin 20^\circ} = 4.
 \end{aligned}$$

$$(ii) \quad \text{Second side} = \sin 18^\circ + \sin 30^\circ = \frac{\sqrt{5}-1}{4} + \frac{1}{2} = \frac{\sqrt{5}+1}{4} = \sin 54^\circ.$$

21. Here $\sin C = \sin B$, $\cos C = -\cos B$.

$\therefore$ Second side $= 2 \cos^2 B = 2(1 - \sin^2 B) = 2(1 - \sin B \sin C)$.

$$\begin{aligned}
 22. \quad \cos^2 B + \cos^2 C &= \frac{1 + \cos 2B}{2} + \frac{1 + \cos 2C}{2} \\
 &= 1 + \frac{1}{2}(\cos 2B + \cos 2C) \\
 &= 1 + \cos(B+C) \cos(B-C) \\
 &= 1 - \cos A \cos(B-C), \text{ for } \cos(B+C) = -\cos A, \\
 &= 1 + \cos^2 A - \cos A \{\cos A + \cos(B-C)\} \\
 &= 1 + \cos^2 A - \cos A \{\cos(B-C) - \cos(B+C)\} \\
 &= 1 + \cos^2 A - 2 \sin B \sin C \cos A.
 \end{aligned}$$

23. As in Ex. 22,

$$\begin{aligned}
 \cos^2 B + \cos^2 C &= 1 + \cos(B+C) \cos(B-C) \\
 &= 1 + \cos A \cos(B-C), \text{ for } \cos(B+C) = \cos A, \\
 &= 1 + \cos^2 A + \cos A \{\cos(B-C) - \cos(B+C)\} \\
 &= 1 + \cos^2 A + 2 \sin B \sin C \cos A.
 \end{aligned}$$

$$\begin{aligned}
 24. \quad \cos^2 A \cos 2B - \cos^2 B \cos 2A &= \cos^2 A (\cos^2 B - \sin^2 B) \\
 &\quad - \cos^2 B (\cos^2 A - \sin^2 A) \\
 &= \cos^2 B \sin^2 A - \cos^2 A \sin^2 B \\
 &= (1 - \sin^2 B) \sin^2 A - (1 - \sin^2 A) \sin^2 B \\
 &= \sin^2 A - \sin^2 B;
 \end{aligned}$$

whence, by transposition, we have the required result.

$$\begin{aligned}
 25. \quad \tan 50^\circ - \tan 40^\circ &= \frac{\sin 50^\circ}{\cos 50^\circ} - \frac{\sin 40^\circ}{\cos 40^\circ} \\
 &= \frac{\sin 50^\circ \cos 40^\circ - \cos 50^\circ \sin 40^\circ}{\cos 50^\circ \cos 40^\circ} \\
 &= \frac{\sin(50^\circ - 40^\circ)}{\cos 50^\circ \cos 40^\circ} = \frac{\sin 10^\circ}{\cos 50^\circ \sin 50^\circ} \\
 &= \frac{2 \sin 10^\circ}{\sin 100^\circ} = \frac{2 \sin 10^\circ}{\cos 10^\circ} = 2 \tan 10^\circ.
 \end{aligned}$$

$$26. \text{ Here } \tan \theta = 2; \therefore \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{4}{1 + 4} = \frac{4}{5} = \cdot 8$$

$$\text{and } \cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - 4}{1 + 4} = -\frac{3}{5} = -\cdot 6.$$

Again, from the Tables, $\cot \theta = \cdot 5$ gives $\theta = 63^\circ 26'$.

$$\therefore 2\theta = 126^\circ 52'.$$

$$\therefore \sin 2\theta = \sin (180^\circ - 126^\circ 52') = \sin 53^\circ 8'$$

$$= \cdot 8, \text{ from the Tables.}$$

$$\cos 2\theta = -\cos 53^\circ 8' = -\cdot 6, \text{ from the Tables.}$$

27. If $\tan \alpha = \cdot 362$, the equation may be written

$$\tan \alpha \cos \theta + \sin \theta = 1, \text{ or } \sin \alpha \cos \theta + \cos \alpha \sin \theta = \cos \alpha.$$

$$\therefore \sin (\alpha + \theta) = \cos \alpha = \sin (90^\circ - \alpha).$$

Now from the Tables, $\alpha = 19^\circ 54'$.

$$\therefore \sin (19^\circ 54' + \theta) = \sin (90^\circ - 19^\circ 54') = \sin 70^\circ 6'.$$

$$\therefore 19^\circ 54' + \theta = 70^\circ 6', \text{ or } 180^\circ - 70^\circ 6';$$

whence

$$\theta = 50^\circ 12', \text{ or } 90^\circ.$$

EXAMPLES. XIII. a. PAGE 128.

$$1. \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{225 + 49 - 169}{210} = \frac{105}{210} = \frac{1}{2}; \therefore C = 60^\circ.$$

$$2. \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{9 + 25 - 49}{80} = -\frac{15}{80} = -\frac{1}{2}; \therefore A = 120^\circ.$$

$$3. \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{5^2 (3 + 1 - 1)}{5^2 2 \sqrt{3}} = \frac{\sqrt{3}}{2}; \therefore A = 30^\circ.$$

Also $a = c$; $\therefore C = A = 30^\circ$; hence $B = 180^\circ - A - C = 120^\circ$.

$$4. \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{961 + 98 - 625}{434 \sqrt{2}} = \frac{1}{\sqrt{2}}; \therefore A = 45^\circ.$$

5. Let $a=2$, $b=2\frac{2}{3}$, $c=3\frac{1}{3}$.

Then
$$\cos C = \frac{4 + \frac{64}{9} - \frac{100}{9}}{\frac{32}{3}} = 0; \therefore C = 90^\circ.$$

6. $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4 + 6 - (4 + 2\sqrt{3})}{4\sqrt{6}} = \frac{3 - \sqrt{3}}{2\sqrt{6}} = \frac{\sqrt{3} - 1}{2\sqrt{2}}; \therefore A = 75^\circ;$

$\cos B = \frac{c^2 + a^2 - b^2}{2ac} = \frac{6 + 4 - 2\sqrt{3} - 4}{2\sqrt{6}(\sqrt{3} + 1)} = \frac{3 + \sqrt{3}}{\sqrt{2}(3 + \sqrt{3})} = \frac{1}{\sqrt{2}}; \therefore B = 45^\circ;$
 $\therefore C = 180^\circ - 75^\circ - 45^\circ = 60^\circ.$

7. $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4 + 4 - 2\sqrt{3} - 2}{4(\sqrt{3} - 1)} = \frac{3 - \sqrt{3}}{2(\sqrt{3} - 1)} = \frac{\sqrt{3}}{2}; \therefore A = 30^\circ;$

$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{4 - 2\sqrt{3} + 2 - 4}{2\sqrt{2}(\sqrt{3} - 1)} = \frac{1 - \sqrt{3}}{\sqrt{2}(\sqrt{3} - 1)} = -\frac{1}{\sqrt{2}};$
 $\therefore B = 135^\circ;$
 $\therefore C = 180^\circ - 30^\circ - 135^\circ = 15^\circ.$

8. $\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{64 + 25 - 19}{80} = \frac{70}{80} = .875; \therefore C = 28^\circ 57'.$

9. $\cos C = \frac{4^2 + 5^2 - 7^2}{2 \times 4 \times 5} = \frac{16 + 25 - 49}{40} = -\frac{8}{40} = -\frac{1}{5} = -\cos 78^\circ 28';$
 $\therefore C = 180^\circ - 78^\circ 28' = 101^\circ 32'.$

10. $c^2 = a^2 + b^2 - 2ab \cos C = 4 + 4 + 2\sqrt{3} - 2 \cdot 2(\sqrt{3} + 1)\frac{1}{2} = 6.$

11. $b^2 = c^2 + a^2 - 2ca \cos B = 9 + 25 - 2 \cdot 3 \cdot 5 \left(-\frac{1}{2}\right) = 9 + 25 + 15 = 49;$
 $\therefore b = 7.$

12. $a^2 = b^2 + c^2 - 2bc \cos A = 49 + 36 - 2 \cdot 7 \cdot 6 \times .2501$
 $= 49 + 36 - 21.0042 = 64, \text{ approx.}; \text{ whence } a = 8.$

13. $a^2 = b^2 + c^2 - 2bc \cos A = 64 + 121 - 2 \cdot 8 \cdot 11 \left(-\frac{1}{16}\right)$
 $= 64 + 121 + 11 = 196;$
 $\therefore a = 14.$

$$14. \quad b^2 = c^2 + a^2 - 2ca \cos B = 9 + 49 - 2 \cdot 3 \cdot 7 (-\cdot 5476) = 81, \text{ approx.};$$

$$\therefore b = 9.$$

$$15. \quad b^2 = c^2 + a^2 - 2ca \cos B = 48 - 24\sqrt{3} + 24 - 2 \cdot 2\sqrt{6} \cdot 2\sqrt{3}(\sqrt{3} - 1) \cdot \frac{\sqrt{3} - 1}{2\sqrt{2}} \\ = 48 - 24\sqrt{3} + 24 - 4 \cdot 3 \cdot (4 - 2\sqrt{3}) = 24;$$

$$\therefore b = 2\sqrt{6}; \text{ whence } A = B = 75^\circ, \text{ and } C = 30^\circ.$$

$$16. \quad a^2 = b^2 + c^2 - 2bc \cos A = 4 + 6 + 2\sqrt{5} - 2 \cdot 2(\sqrt{5} + 1) \cdot \frac{\sqrt{5} - 1}{4} \\ = 4 + 6 + 2\sqrt{5} - 4 = 6 + 2\sqrt{5};$$

$$\therefore a = \sqrt{5} + 1; \text{ whence } C = A = 72^\circ, \text{ and } B = 36^\circ.$$

$$17. \quad C = 75^\circ; \text{ whence } a = c.$$

Also
$$a = \frac{b \sin A}{\sin B} = \frac{\sqrt{8}(\sqrt{3} + 1)2}{2\sqrt{2}} = 2\sqrt{3} + 2.$$

$$18. \quad c = \frac{b \sin C}{\sin B} = \sqrt{6} \cdot \frac{\sqrt{3} - 1}{2\sqrt{2}} \cdot \frac{2}{\sqrt{3}} = \sqrt{3} - 1.$$

Also
$$A = 105^\circ; \text{ whence } a = \frac{b \sin A}{\sin B} = \sqrt{6} \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}} \cdot \frac{2}{\sqrt{3}} = \sqrt{3} + 1.$$

$$19. \quad C = 30^\circ; \text{ whence } a = \frac{c \sin A}{\sin C} = \frac{\sqrt{2} \times 2}{\sqrt{2}} = 2;$$

$$b = \frac{a \sin B}{\sin A} = 2 \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}} \cdot \sqrt{2} = \sqrt{3} + 1.$$

$$20. \quad \text{Here } \frac{c}{a} = \frac{\sin C}{\sin A} = \frac{\sin 75^\circ}{\sin 45^\circ} = \frac{\sqrt{3} + 1}{2\sqrt{2}} \bigg/ \frac{1}{\sqrt{2}} = \frac{\sqrt{3} + 1}{2}.$$

$$21. \quad \sin A = \frac{a \sin C}{c} = \frac{2}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{2};$$

$$\therefore b = a \cos C + c \cos A = 2 \left(-\frac{1}{2} \right) + 2\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 3 - 1 = 2.$$

$$22. \quad \sin A = \frac{a \sin B}{b} = \frac{3}{3\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{2};$$

$$\therefore c = a \cos B + b \cos A = 3 \cdot \frac{1}{2} + 3\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 6.$$

$$23. \quad \text{We have } (b+c)^2 - a^2 = 3bc;$$

$$\therefore \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{2}; \text{ whence } A = 60^\circ.$$

$$24. \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{12 + 6 - (12 + 6\sqrt{3})}{2 \cdot 2\sqrt{3} \cdot \sqrt{6}} \\ = \frac{6(1 - \sqrt{3})}{12\sqrt{2}} = -\frac{\sqrt{3} - 1}{2\sqrt{2}} = -\cos 75^\circ;$$

$$\therefore A = 105^\circ; \text{ similarly } C = 30^\circ.$$

$$\therefore \sin B = \frac{b \sin C}{c} = \frac{2\sqrt{3}}{2\sqrt{6}} = \frac{1}{\sqrt{2}}; \text{ whence } B = 45^\circ.$$

$$25. \quad \text{The sides are proportional to } \sqrt{3} + 1, \sqrt{3} - 1, \sqrt{6};$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4 - 2\sqrt{3} + 6 - (4 + 2\sqrt{3})}{2\sqrt{6}(\sqrt{3} - 1)} \\ = \frac{6 - 4\sqrt{3}}{2\sqrt{6}(\sqrt{3} - 1)} = \frac{2\sqrt{3}(\sqrt{3} - 2)}{2\sqrt{3} \cdot \sqrt{2}(\sqrt{3} - 1)} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} \\ = \frac{1 - \sqrt{3}}{2\sqrt{2}} = -\cos 75^\circ;$$

$$\therefore A = 105^\circ.$$

$$\sin C = \frac{c}{a} \sin A = \frac{\sqrt{6}}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}} = \frac{\sqrt{3}}{2};$$

$$\therefore C = 60^\circ, \text{ and } B = 15^\circ.$$

$$26. \quad \text{Here } b = \frac{\sqrt{6} + \sqrt{2}}{4}, \quad c = \frac{\sqrt{6} - \sqrt{2}}{4}, \quad A = 60^\circ;$$

$$\therefore a^2 = \frac{(\sqrt{6} + \sqrt{2})^2 + (\sqrt{6} - \sqrt{2})^2}{16} - \frac{2 \cdot 4}{16} \cdot \frac{1}{2} = \frac{12}{16} = \frac{3}{4};$$

$$\therefore a = \frac{\sqrt{3}}{2}.$$

$$\sin C = \frac{c}{a} \sin A = \frac{\sqrt{6} - \sqrt{2}}{4} \cdot \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}};$$

whence

$$C = 15^\circ, \text{ and } B = 105^\circ.$$

EXAMPLES. XIII. b. PAGE 132.

$$1. \quad \sin B = \frac{b \sin A}{a} = \sqrt{3} \times \frac{1}{2} = \frac{\sqrt{3}}{2}.$$

$\therefore B = 60^\circ$ or 120° ; and since $a < b$, both these values are admissible. Hence $C = 90^\circ$ or 30° .

$$c = \frac{a \sin C}{\sin A} = 2 \text{ or } 1, \text{ on reduction.}$$

$$2. \quad \sin B = \frac{b \sin C}{c} = \frac{3\sqrt{2}}{2\sqrt{3}} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2}.$$

$\therefore B = 60^\circ$ or 120° ; and since $c < b$, both these values are admissible. Hence $A = 75^\circ$ or 15° .

$$a = \frac{b \sin A}{\sin B} = 3 + \sqrt{3}, \text{ or } 3 - \sqrt{3}, \text{ on reduction.}$$

$$3. \quad \sin A = \frac{a \sin C}{c} = \frac{2}{\sqrt{6}} \times \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}.$$

$\therefore A = 45^\circ$, the other value being inadmissible, since $c > a$. Hence $B = 75^\circ$.

$$b = \frac{a \sin B}{\sin A} = \sqrt{3} + 1.$$

4. $\sin C = \frac{c}{a} \sin A = \frac{5}{2} \times \frac{1}{2} = \frac{5}{4}$, which is impossible. Thus there is no triangle with the given parts.

$$5. \quad \sin C = \frac{c}{b} \sin B = \frac{2\sqrt{3}}{\sqrt{6}} \cdot \frac{1}{2} = \frac{1}{\sqrt{2}}.$$

$\therefore C = 45^\circ$ or 135° ; and since $b < c$, both values are admissible. Hence $A = 105^\circ$ or 15° .

$$a = \frac{b \sin A}{\sin B} = \sqrt{3}(\sqrt{3} + 1), \text{ or } \sqrt{3}(\sqrt{3} - 1), \text{ on reduction.}$$

$$6. \quad \sin C = \frac{c}{b} \sin B = \frac{3 + \sqrt{3}}{3\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}.$$

$\therefore C = 75^\circ$ or 105° ; and since $b < c$, both values are admissible. Hence $A = 45^\circ$ or 15° .

$$a = \frac{b \sin A}{\sin B} = 2\sqrt{3}, \text{ or } 3 - \sqrt{3}, \text{ on reduction.}$$

$$7. \quad \sin A = \frac{a}{c} \sin C = \frac{3 + \sqrt{3}}{3 - \sqrt{3}} \cdot \frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{\sqrt{3} + 1}{2\sqrt{2}}.$$

$\therefore A = 75^\circ$ or 105° ; and since $c < a$, both values are admissible. Hence $B = 90^\circ$ or 60° .

$$b = \frac{c \sin B}{\sin C} = 2\sqrt{6}, \text{ or } 3\sqrt{2}, \text{ on reduction.}$$

$$8. \sin B = \frac{b}{a} \sin A = \frac{4(\sqrt{5}+1)}{4} \cdot \frac{\sqrt{5}-1}{4} = 1.$$

$\therefore B = 90^\circ$, and there is no ambiguity.

$$c^2 = b^2 - a^2 = (b+a)(b-a) = (8+\sqrt{80})(8-\sqrt{80})$$

$$= 4(2+\sqrt{5}) \cdot 4\sqrt{5} = 16\sqrt{5}(2+\sqrt{5})$$

$$\therefore c = 4\sqrt{5+2\sqrt{5}}.$$

$$9. \sin A = \frac{a}{b} \sin B = \frac{3\sqrt{2}}{2\sqrt{3}} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2}.$$

$\therefore A = 60^\circ$ or 120° ; but *neither* of these values is admissible as in each case the sum of the angles would be greater than 180° . Thus the triangle is impossible.

EXAMPLES. XIII. c. PAGE 134.

1. Follows at once from Art. 137.

2. The first side $= b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2 = a^2 + b^2 + c^2$.

3. The first side $= \frac{a^2 + b^2 - c^2}{2} - \frac{a^2 + c^2 - b^2}{2} = b^2 - c^2$.

4. The first side $= (b \cos A + a \cos B) + (c \cos B + b \cos C)$
 $+ (c \cos A + a \cos C) = c + a + b$.

5. The first side $= a(1 - \cos C) + c(1 - \cos A)$
 $= a + c - (a \cos C + c \cos A)$
 $= a + c - b$.

6. The second side $= \frac{a \cos B}{a \cos C} = \frac{\cos B}{\cos C}$.

7. The second side $= \frac{a \sin C}{c \cos A} = \frac{c \sin A}{c \cos A} = \tan A$.

8. Put $k = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$; then

$$\begin{aligned} \text{the first side} &= k(\sin B + \sin C) \sin \frac{A}{2} \\ &= 2k \sin \frac{B+C}{2} \cdot \cos \frac{B-C}{2} \cdot \sin \frac{A}{2} \\ &= 2k \sin \frac{A}{2} \cos \frac{A}{2} \cdot \cos \frac{B-C}{2} \\ &= k \sin A \cdot \cos \frac{B-C}{2} \\ &= a \cos \frac{B-C}{2}. \end{aligned}$$

$$\begin{aligned}
 9. \quad \text{The first side} &= \frac{k (\sin A + \sin B)}{k \sin C} \sin^2 \frac{C}{2} \\
 &= \frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \sin \frac{C}{2} \cos \frac{C}{2}} \sin^2 \frac{C}{2} \\
 &= \cos \frac{A-B}{2} \sin \frac{C}{2} \\
 &= \cos \frac{A-B}{2} \cos \frac{A+B}{2} \\
 &= \frac{\cos A + \cos B}{2}.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \text{The first side} &= k \{ \sin A \sin (B-C) + \dots + \dots \} \\
 &= k \{ \sin (B+C) \sin (B-C) + \dots + \dots \} \\
 &= k \{ \sin^2 B - \sin^2 C + \dots + \dots \} = 0.
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \text{The second side} &= \frac{\sin^2 A - \sin^2 B}{\sin^2 C} \\
 &= \frac{\sin (A+B) \sin (A-B)}{\sin (A+B) \sin (A+B)} = \frac{\sin (A-B)}{\sin (A+B)}.
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \text{The second side} &= \frac{\sin^2 A - \sin^2 B}{\sin^2 C - \sin^2 A} = \frac{\sin (A+B) \sin (A-B)}{\sin (C+A) \sin (C-A)} \\
 &= \frac{\sin C \sin (A-B)}{\sin B \sin (C-A)} = \frac{c \sin (A-B)}{b \sin (C-A)}.
 \end{aligned}$$

EXAMPLES. XIII. d. PAGE 136.

1. Let ABC be the triangle, in which $B=C=2A$, and $a=2$;

then $B+C+A=5A=180^\circ$; $\therefore A=36^\circ$, $B=C=72^\circ$;

and $b=c=\frac{a}{\sin A} \sin B = \frac{a \sin 2A}{\sin A} = 2a \cos A = 4 \cdot \frac{\sqrt{5}+1}{4} = \sqrt{5}+1$.

2. $A=180^\circ - B - C=60^\circ$;

If AD be the perpendicular, then $c=AD \operatorname{cosec} 45^\circ = 3\sqrt{2}$,

$$b=AD \operatorname{cosec} 75^\circ = \frac{6\sqrt{2}}{\sqrt{3}+1} = 3(\sqrt{6}-\sqrt{2})$$

$$a=BD+DC=AD \cot 45^\circ + AD \cot 75^\circ = 3+3(2-\sqrt{3})=9-3\sqrt{3}.$$

$$3. \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{28 - 16\sqrt{3} + 24 - 12\sqrt{3} - 4}{4\sqrt{2}(9 - 5\sqrt{3})} = \frac{48 - 28\sqrt{3}}{4\sqrt{2}(9 - 5\sqrt{3})}$$

$$= \frac{(12 - 7\sqrt{3})(9 + 5\sqrt{3})}{\sqrt{2} \cdot 6} = \frac{3 - 3\sqrt{3}}{6\sqrt{2}} = -\frac{\sqrt{3} - 1}{2\sqrt{2}};$$

$$\therefore A = 180^\circ - 75^\circ = 105^\circ.$$

$$\sin C = \frac{c}{a} \sin A = \frac{3\sqrt{2} - \sqrt{6}}{2} \cdot \frac{\sqrt{3} + 1}{2\sqrt{2}} = \frac{\sqrt{3}}{2} \cdot \frac{(\sqrt{3} - 1)(\sqrt{3} + 1)}{2} = \frac{\sqrt{3}}{2};$$

$$\therefore C = 60^\circ;$$

and $B = 180^\circ - 105^\circ - 60^\circ = 15^\circ.$

4. We have

$$b - a = 2, \quad ab = 4;$$

$$\therefore b + a = 2\sqrt{5}, \text{ rejecting the negative sign,}$$

$$\therefore a = \sqrt{5} - 1, \quad b = \sqrt{5} + 1;$$

$$\therefore \sin B = \frac{b \sin A}{a} = \frac{\sqrt{5} + 1}{\sqrt{5} - 1} \cdot \frac{\sqrt{5} - 1}{4} = \frac{\sqrt{5} + 1}{4};$$

$$\therefore B = 54^\circ \text{ or } 126^\circ;$$

$$\therefore C = 108^\circ \text{ or } 36^\circ.$$

$$5. \sin c = \frac{c \sin B}{b} = 150 \cdot \frac{1}{2} \cdot \frac{1}{50\sqrt{3}} = \frac{\sqrt{3}}{2};$$

$$\therefore C = 60^\circ \text{ or } 120^\circ; \text{ both values being admissible since } b < c;$$

$$\therefore A = 90^\circ \text{ or } 30^\circ.$$

In the first case, $a = \sin A \cdot \frac{b}{\sin B} = 50\sqrt{3} \times 2 = 100\sqrt{3}.$

In the second case, $a = b = 50\sqrt{3}.$

If $B = 30^\circ, C = 150^\circ, b = 75^\circ;$

$$\sin C = \frac{150}{75} \times \sin 30^\circ = 1;$$

$$\therefore C = 90^\circ, \text{ and there is no ambiguity.}$$

6. We have at once from a figure, $\sin B = \frac{\sqrt{5} - 1}{4};$ whence $B = 18^\circ;$

$$\therefore C = 180^\circ - 36^\circ - 18^\circ = 126^\circ.$$

7. Let ABC be the triangle, and let $B = 22\frac{1}{2}^\circ, C = 112\frac{1}{2}^\circ,$ and let AD be the perpendicular from A on $BC.$ Then $A = 180^\circ - 22\frac{1}{2}^\circ - 112\frac{1}{2}^\circ = 45^\circ;$ also

$$AD = AB \sin 22\frac{1}{2}^\circ,$$

$$BC = \frac{AB}{\sin 112\frac{1}{2}^\circ} \cdot \sin 45^\circ = \frac{AB}{\sqrt{2} \cos 22\frac{1}{2}^\circ} = \frac{2AB \sin 22\frac{1}{2}^\circ}{\sqrt{2} \sin 45^\circ} = 2AD.$$

That is, the altitude is half the base.

$$8. \quad \frac{\sin A}{\sin B} = \frac{a}{b} = 2; \text{ and } \sin A = 2 \sin B = \sin 3B;$$

$$\therefore \frac{3 \sin B - 4 \sin^3 B}{\sin B} = 2, \text{ or } 3 - 4 \sin^2 B = 2.$$

$$\therefore \sin B = \frac{1}{2}, \text{ rejecting the negative value.}$$

$$\therefore B = 30^\circ, A = 90^\circ, C = 60^\circ.$$

$$\text{Also} \quad c = a \sin C = \frac{a \sqrt{3}}{2}.$$

9. Let C be the greatest angle, then

$$\begin{aligned} \cos C &= \frac{(2x+3)^2 + (x^2+2x)^2 - (x^2+3x+3)^2}{2(2x+3)(x^2+2x)} \\ &= \frac{-2x^3 - 7x^2 - 6x}{2(2x+3)(x^2+2x)} = -\frac{(2x+3)(x+2)}{2(2x+3)(x+2)} = -\frac{1}{2}. \end{aligned}$$

Thus the greatest angle is 120° .

$$10. \quad \text{First side} = b \cos C + c \cos B - a \cos C - c \cos A = a - b.$$

$$\text{Now} \quad \frac{a-b}{c} = \frac{\sin A - \sin B}{\sin C} = \frac{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}}{2 \sin \frac{C}{2} \cos \frac{C}{2}}$$

$$= \frac{\sin \frac{A-B}{2}}{\cos \frac{C}{2}} = \sin \frac{A-B}{2} \operatorname{cosec} \frac{A+B}{2}.$$

$$11. \quad \frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A} = \frac{\sin B + \sin(A+B)}{\sin A} = \frac{2 \sin \left(B + \frac{A}{2}\right) \cos \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}}.$$

$$\therefore (b+c) \sin \frac{A}{2} = a \sin \left(\frac{A}{2} + B\right).$$

$$12. \quad \frac{a+b}{b+c} = \frac{\sin A + \sin B}{\sin B + \sin C} = \frac{\sin \overline{B+C} + \sin B}{\sin B + \sin C} = \frac{2 \sin \left(B + \frac{C}{2}\right) \cos \frac{C}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}}$$

$$= \frac{\sin \left(B + \frac{C}{2}\right) \cos \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B-C}{2}}.$$

$$\begin{aligned}
 13. \text{ First side} &= \frac{1 - \cos(A - B) \cos(A + B)}{1 - \cos(A - C) \cos(A + C)} = \frac{1 - (\cos^2 A - \sin^2 B)}{1 - (\cos^2 A - \sin^2 C)} \\
 &= \frac{\sin^2 A + \sin^2 B}{\sin^2 A + \sin^2 C} = \frac{a^2 + b^2}{a^2 + c^2}.
 \end{aligned}$$

$$\begin{aligned}
 14. \text{ We have } &c^4 - 2(a^2 + b^2)c^2 + a^4 + a^2b^2 + b^4 = 0; \\
 &\therefore (c^2 - a^2 + ab + b^2)(c^2 - a^2 - ab + b^2) = 0; \\
 &\therefore c^2 = a^2 + ab + b^2, \text{ or } a^2 - ab + b^2;
 \end{aligned}$$

$$\begin{aligned}
 \text{But} \quad &c^2 = a^2 + b^2 + 2ab \cos C; \\
 &\therefore 2 \cos C = 1, \text{ or } -1; \\
 &\therefore C = 60^\circ, \text{ or } 120^\circ.
 \end{aligned}$$

15. See figure of the Ambiguous Case on page 131.

$$(1) \quad c_1 - c_2 = B_1 B_2 = 2B_1 D = 2a \cos B_1.$$

$$(2) \quad \cos \frac{C_1 - C_2}{2} = \cos B_1 CD = \frac{CD}{CB_1} = \frac{b \sin A}{a}.$$

(3) c_1, c_2 are the roots of the quadratic

$$c^2 - 2b \cos A \cdot c + b^2 - a^2 = 0; \quad [\text{Art. 150.}]$$

$$\therefore c_1 + c_2 = 2b \cos A; \quad c_1 c_2 = b^2 - a^2.$$

$$\begin{aligned}
 \therefore c_1^2 + c_2^2 - 2c_1 c_2 \cos 2A &= c_1^2 + c_2^2 - 2c_1 c_2 (2 \cos^2 A - 1) \\
 &= (c_1 + c_2)^2 - 4(b^2 - a^2) \cos^2 A \\
 &= 4b^2 \cos^2 A - 4b^2 \cos^2 A + 4a^2 \cos^2 A \\
 &= 4a^2 \cos^2 A.
 \end{aligned}$$

$$(4) \text{ We have } \quad C_1 + C_2 = 2 \angle ACD;$$

$$C_1 - C_2 = 2 \angle B_1 CD;$$

$$\begin{aligned}
 \therefore \sin \frac{C_1 + C_2}{2} \sin \frac{C_1 - C_2}{2} &= \sin ACD \sin B_1 CD \\
 &= \cos CAB_1 \cos CB_1 A \\
 &= \cos A \cos B.
 \end{aligned}$$

16. See figure of Art. 148 (iii). We have $A = 45^\circ$; $\therefore \angle ACD = 45^\circ$.

$$\text{Hence} \quad DC = DA = \frac{c_1 + c_2}{2}; \quad \text{also} \quad DB_2 = \frac{c_1 - c_2}{2};$$

$$\therefore CB_2^2 = \left(\frac{c_1 + c_2}{2}\right)^2 + \left(\frac{c_1 - c_2}{2}\right)^2 = \frac{c_1^2 + c_2^2}{2};$$

$$\therefore \cos^2 B_2 CD = \frac{CD^2}{CB_2^2} = \frac{(c_1 + c_2)^2}{2(c_1^2 + c_2^2)};$$

$$\therefore \cos B_1 CB_2 = 2 \cos^2 B_2 CD - 1 = \frac{(c_1 + c_2)^2}{c_1^2 + c_2^2} - 1 = \frac{2c_1 c_2}{c_1^2 + c_2^2}.$$

17. From the given condition we have

$$\sin C \cos A + 2 \cos C \sin C = \sin B \cos A + 2 \cos B \sin B,$$

$$\text{or} \quad \cos A (\sin C - \sin B) = \sin 2B - \sin 2C.$$

This easily reduces to

$$\cos A \sin \frac{B-C}{2} \cos \frac{B+C}{2} = 2 \cos A \sin \frac{B-C}{2} \cos \frac{B-C}{2}.$$

Now $\cos \frac{B+C}{2}$ cannot equal $2 \cos \frac{B-C}{2}$; hence we must have

$$\cos A = 0, \text{ which gives } A = 90^\circ;$$

$$\text{or} \quad \sin \frac{B-C}{2} = 0, \text{ in which case } B = C.$$

18. Since a, b, c are in A.P.; we have $a - b = b - c$;

$$\therefore \sin A - \sin B = \sin B - \sin C;$$

$$\therefore 2 \sin \frac{A-B}{2} \cos \frac{A+B}{2} = 2 \sin \frac{B-C}{2} \cos \frac{B+C}{2};$$

$$\therefore \frac{\sin \frac{A-B}{2}}{\sin \frac{A}{2} \sin \frac{B}{2}} = \frac{\sin \frac{B-C}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}};$$

$$\therefore \cot \frac{B}{2} - \cot \frac{A}{2} = \cot \frac{C}{2} - \cot \frac{B}{2};$$

That is $\cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2}$ are in A.P.

19. Let $k = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$; then

$$\begin{aligned} \text{first side} &= \frac{k^2 \sin A \sin \overline{B+C} \sin \overline{B-C}}{\sin B + \sin C} + \dots + \dots \\ &= k^2 \left[\frac{\sin A (\sin^2 B - \sin^2 C)}{\sin B + \sin C} + \dots + \dots \right] \\ &= k^2 [\sin A (\sin B - \sin C) + \dots + \dots] = 0. \end{aligned}$$

MISCELLANEOUS EXAMPLES. D. PAGE 138.

$$1. \quad (1) \quad \tan 2\theta \cot \theta - 1 = \frac{2}{1 - \tan^2 \theta} - 1 = \frac{1 + \tan^2 \theta}{1 - \tan^2 \theta} = \sec 2\theta.$$

$$(2) \quad \sin \alpha - \cot \theta \cos \alpha = \frac{\sin \alpha \sin \theta - \cos \alpha \cos \theta}{\sin \theta} = -\operatorname{cosec} \theta \cos (\alpha + \theta).$$

$$\begin{aligned}
 2. \quad c^2 &= a^2 + b^2 - 2ab \cos C = 48^2 + 35^2 - 48 \times 35 \\
 &= 13^2 + 48 \times 35 = 1849; \\
 \therefore c &= 43.
 \end{aligned}$$

$$3. \quad \text{Here } \tan \alpha = \sqrt{\left(\frac{17}{8}\right)^2 - 1} = \frac{15}{8};$$

$$\text{also} \quad \tan \beta = \sqrt{\left(\frac{17}{15}\right)^2 - 1} = \frac{8}{15} = \cot \alpha;$$

$$\therefore \alpha + \beta = 90^\circ;$$

$$\therefore \tan(\alpha + \beta) = \infty, \text{ and } \operatorname{cosec}(\alpha + \beta) = 1.$$

$$4. \quad \text{The expression} = \frac{2 \sin 8a \cos 15a}{2 \sin 8a \cos 6a} = \frac{\cos 15a}{\cos 6a} = -1,$$

since $15a = \pi - 6a$.

$$\begin{aligned}
 5. \quad \text{First side} &= \frac{1}{2} (\sin 3\theta - \sin \theta + \sin 5\theta - \sin 3\theta + \sin 7\theta - \sin 5\theta) \\
 &= \frac{1}{2} (\sin 7\theta - \sin \theta) \\
 &= \sin 3\theta \cos 4\theta.
 \end{aligned}$$

$$6. \quad a^2 = 2 + 4 + 2\sqrt{3} - 2(\sqrt{3} + 1) = 4; \text{ whence } a = 2.$$

$$\text{Again} \quad \sin B = \frac{b \sin A}{a} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2};$$

$\therefore B = 30^\circ$, or 150° ; but the latter value is inadmissible since c is the greatest side. Therefore $C = 180^\circ - (45^\circ + 30^\circ) = 105^\circ$.

$$\begin{aligned}
 7. \quad (1) \quad 2 \sin^2 36^\circ &= 1 - \cos 72^\circ = 1 - \sin 18^\circ \\
 &= 1 - \frac{\sqrt{5} - 1}{4} = \frac{5 - \sqrt{5}}{4} = \sqrt{5} \sin 18^\circ.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad 4 \sin 36^\circ \cos 18^\circ &= 2 (\sin 54^\circ + \sin 18^\circ) \\
 &= \frac{\sqrt{5} + 1}{2} + \frac{\sqrt{5} - 1}{2} = \sqrt{5}.
 \end{aligned}$$

$$8. \quad \frac{\sin 3a}{\sin a} + \frac{\cos 3a}{\cos a} = \frac{\sin 3a \cos a + \cos 3a \sin a}{\sin a \cos a} = \frac{2 \sin 4a}{\sin 2a} = 4 \cos 2a.$$

$$9. \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4 + 4 - 8 + 4\sqrt{3}}{8} = \frac{\sqrt{3}}{2};$$

$$\therefore A = 30^\circ.$$

$$\therefore B = C = \frac{1}{2} (180^\circ - 30^\circ) = 75^\circ.$$

10. (1) Each expression easily reduces to $\frac{\sin a}{\sin 3a}$.

$$\begin{aligned}(2) \quad \cos a + \cos 2a + \cos 3a &= 2 \cos \frac{3a}{2} \cos \frac{a}{2} + 2 \cos^2 \frac{3a}{2} - 1 \\ &= 2 \cos \frac{3a}{2} \left(\cos \frac{a}{2} + \cos \frac{3a}{2} \right) - 1 \\ &= 4 \cos a \cos \frac{a}{2} \cos \frac{3a}{2} - 1.\end{aligned}$$

$$\begin{aligned}11. (1) \quad \text{First side} &= 2b^2 \sin C \cos C + 2c^2 \sin B \cos B \\ &= 2b \sin C (b \cos C + c \cos B) \\ &= 2ab \sin C = 2bc \sin A.\end{aligned}$$

$$\begin{aligned}(2) \quad \text{First side} &= k (\sin A \sin \overline{B-C} + \dots + \dots) \\ &= k (\sin B + C \sin \overline{B-C} + \dots + \dots) \\ &= k (\sin^2 B - \sin^2 C + \dots + \dots) = 0.\end{aligned}$$

$$12. \quad \tan(A+B) = \tan(360^\circ - \overline{C+D});$$

$$\therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} = - \frac{\tan C + \tan D}{1 - \tan C \tan D}.$$

$$\therefore \tan A + \tan B + \tan C + \tan D = \tan C \tan D (\tan A + \tan B) + \tan A \tan B (\tan C + \tan D);$$

$$\text{or } \tan A + \tan B + \tan C + \tan D$$

$$= \tan A \tan B \tan C \tan D (\cot A + \cot B + \cot C + \cot D).$$

EXAMPLES. XIV. a. PAGE 145.

For Examples 1—3 see Arts. 151, 152.

For Examples 4—7 see Arts. 162, 163.

$$8. \quad \log 768 = \log (2^8 \times 3) = 8 \log 2 + \log 3 = 2.8853613.$$

$$9. \quad \log 2352 = \log (2^4 \times 3 \times 7^2) = 4 \log 2 + \log 3 + 2 \log 7 = 3.3714373.$$

$$\begin{aligned}10. \quad \log 35.28 &= \log \left(\frac{2^3 \times 3^2 \times 7^2}{10^2} \right) = 3 \log 2 + 2 (\log 3 + \log 7 - 1) \\ &= .90309 + 2 (.4771213 + .845098 - 1) \\ &= .90309 + .6444386 = 1.5475286.\end{aligned}$$

$$\begin{aligned}11. \quad \log \sqrt{6804} &= \frac{1}{2} \log (2^2 \times 3^5 \times 7) = \frac{1}{2} (2 \log 2 + 5 \log 3 + \log 7) \\ &= \frac{1}{2} (.60206 + 2.3856065 + .845098) \\ &= 1.9163822.\end{aligned}$$

$$\begin{aligned}
 12. \quad \log \sqrt[5]{\cdot 00162} &= \frac{1}{5} \log \left(\frac{2 \times 3^4}{10^5} \right) = \frac{1}{5} (\log 2 + 4 \log 3 - 5) \\
 &= \frac{1}{5} (\bar{3} \cdot 2095152) = \bar{1} \cdot 441903.
 \end{aligned}$$

$$\begin{aligned}
 13. \quad \log \cdot 0217 &= \log \frac{217 - 21}{9000} = \log \frac{196}{9000} = \log \frac{7^2}{15^2 \times 10} \\
 &= 2 (\log 7 - \log 3 - \log 5) - 1 \\
 &= 2 (\cdot 8450980 - 1 \cdot 1760913) - 1 = \bar{2} \cdot 3380134.
 \end{aligned}$$

$$14. \quad \log \cos 60^\circ = \log \left(\frac{1}{2} \right) = -\log 2 = -\cdot 30103 = \bar{1} \cdot 69897.$$

$$\begin{aligned}
 15. \quad \log \sin^3 60^\circ &= 3 \log \left(\frac{\sqrt{3}}{2} \right) = \frac{3}{2} \log 3 - 3 \log 2 \\
 &= \cdot 7156819 - \cdot 90309 = \bar{1} \cdot 8125919.
 \end{aligned}$$

$$16. \quad \log \sqrt[3]{\sec 45^\circ} = \frac{1}{3} \log \sqrt{2} = \frac{1}{6} \log 2 = \cdot 0501716.$$

$$\begin{aligned}
 17. \quad \text{The expression} &= \log \left[\left(\frac{15}{8} \right)^2 \times \frac{162}{25} \times \left(\frac{4}{9} \right)^3 \right] \\
 &= \log \left(\frac{15 \times 15 \times 81 \times 2 \times 64}{64 \times 25 \times 81 \times 9} \right) = \log 2.
 \end{aligned}$$

$$\begin{aligned}
 18. \quad \text{The expression} \\
 &= 16 (1 - 2 \log 3) - 4 (2 \log 5 - 3 \log 2 - \log 3) - 7 (3 \log 2 + 1 - 4 \log 3) \\
 &= 9 - 9 \log 2 - 8 \log 5 = 1 - \log 2 = \cdot 69897.
 \end{aligned}$$

$$19. \quad \log x = \frac{1}{7} \log 7 = \cdot 1207283. \quad \therefore x = 1 \cdot 320469.$$

$$\begin{aligned}
 20. \quad \log x &= \frac{1}{3} \log \left(\frac{2^2 \times 3^2 \times 7^2}{10^3} \right) = \frac{1}{3} (2 \log 2 + 2 \log 3 + 2 \log 7 - 8) \\
 &= \frac{1}{3} (3 \cdot 2464986 - 8) = \bar{2} \cdot 4154995. \\
 \therefore x &= \cdot 0260315.
 \end{aligned}$$

$$21. \quad \log x = \cdot 5527899 + \bar{2} \cdot 5527899 + 1 \cdot 1842633 = \cdot 2898431.$$

$$\begin{aligned}
 22. \quad \log x &= \frac{1}{3} \log \frac{11}{10^5} + 2 \log \frac{11^2}{10^2} + \frac{4}{3} \log \frac{11^3}{10^2} - \log (11^2 \times 10^5) \\
 &= \frac{1}{3} (\log 11 - 5) + 4 (\log 11 - 1) + \frac{4}{3} (3 \log 11 - 2) - 2 \log 11 - 5. \\
 &= \frac{19}{3} \log 11 - \frac{40}{3} = \frac{1}{3} (21 \cdot 7864613) = \bar{7} \cdot 2621538.
 \end{aligned}$$

23. Since $\left(\frac{21}{20}\right)^{300} = \left(\frac{3 \times 7}{2^2 \times 5}\right)^{300}$;

$$\begin{aligned}\therefore \log \left(\frac{21}{20}\right)^{300} &= 300 (\log 3 + \log 7 - 1 - \log 2) \\ &= 396.66579 - 390.30900 = 6.35679;\end{aligned}$$

$\therefore \left(\frac{21}{20}\right)^{300}$ has 7 digits in its integral part.

Since $\left(\frac{126}{125}\right)^{1000} = \left(\frac{2 \times 3^2 \times 7}{5^3}\right)^{1000}$;

$$\begin{aligned}\therefore \log \left(\frac{126}{125}\right)^{1000} &= 1000 (\log 2 + 2 \log 3 + \log 7 - 3 + 3 \log 2) \\ &= 1000 (4 \log 2 + 2 \log 3 + \log 7 - 3) \\ &= 3.4606;\end{aligned}$$

$\therefore \left(\frac{126}{125}\right)^{1000}$ has 4 digits in its integral part.

24. 7^4 is the smallest number whose logarithm has characteristic 4.

7^3 is the smallest number whose logarithm has characteristic 3.

$\therefore$ the required number is $7^4 - 7^3 = 2058$.

EXAMPLES. XIV. b. PAGE 149.

$$\begin{aligned}1. \log x &= \frac{2}{3} \log \left(\frac{147 \times 375}{126 \times 16}\right) = \frac{2}{3} \log \left(\frac{7 \times 5^3}{2^5}\right) \\ &= \frac{2}{3} \log \frac{7 \times 1000}{16 \times 16} = \frac{2}{3} (3 + \log 7 - 8 \log 2) \\ &= \frac{2}{3} (3.8450980 - 2.4082400) = \frac{2}{3} (1.4368580) = .9579053; \\ \therefore x &= 9.076226.\end{aligned}$$

$$\begin{aligned}2. \log x &= \frac{1}{3} \log (3^3 \times 2 \times 7) + \frac{1}{2} \log (3^3 \times 2^2) \\ &\quad - \frac{1}{6} \log (3^3 \times 7 \times 2^4) - \frac{1}{3} \log (2 \times 3^5) \\ &= \frac{1}{2} \log 3 + \frac{1}{3} \log 2 + \frac{1}{6} \log 7 = .4797536; \\ \therefore x &= 3.01824.\end{aligned}$$

.1003433
.2385606
.1408497
+4797536

$$\begin{aligned}
 3. \quad \log x &= \frac{1}{2} \log (10 \times 2^2 \times 3^3) + \frac{5}{3} \log (2^3 \times 3 \div 100) + \log (10 \times 3^4) \\
 &= \frac{43}{6} \log 3 + 6 \log 2 - \frac{11}{6} = \frac{1}{6} (20 \cdot 5162159 - 11) + 1 \cdot 8061800 \\
 &= 1 \cdot 5860360 + 1 \cdot 8061800 = 3 \cdot 3922160; \\
 &\therefore x = 2467 \cdot 266.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \text{Let } x \text{ be the value; then } 20^x &= 800. \\
 \therefore x \log 20 &= \log 800; \\
 \therefore x &= \frac{\log 800}{\log 20} = \frac{2 + 3 \log 2}{1 + \log 2} = 2 \cdot 23.
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \text{Here } 3^x &= 49, \text{ or } x \log 3 = 2 \log 7. \\
 \therefore x &= \frac{2 \log 7}{\log 3} = 3 \cdot 54.
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \text{Here } 125^x &= 4000, \text{ or } x \log 5^3 = \log (2^2 \times 10^3). \\
 \therefore x &= \frac{3 + 2 \log 2}{3 \log 5} = 1 \cdot 72.
 \end{aligned}$$

$$\begin{aligned}
 7. \quad \log x &= \frac{40}{3} \log \left(\frac{378}{10^5} \right) = \frac{40}{3} (\log 2 + 3 \log 3 + \log 7 - 5) = \overline{33} \cdot 699892. \\
 \therefore \text{the number of ciphers is } 32.
 \end{aligned}$$

$$\begin{aligned}
 \log x &= 50 \log \frac{259}{9990} = 50 \log \frac{7}{270} \\
 &= 50 (\log 7 - 3 \log 3 - 1) = 50 (\cdot 8450980 - 2 \cdot 4313639) \\
 &= 50 (\overline{2} \cdot 4137341) = \overline{80} \cdot 6867050. \\
 \therefore \text{the number of ciphers is } 79.
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \text{Let } x \text{ be the base; then } x^3 &= 11000; \\
 \therefore 3 \log x &= \log 11 + 3 = 4 \cdot 0413927; \\
 \therefore \log x &= 1 \cdot 3471309. \\
 \text{But } \log 222398 &= 5 \cdot 3471309; \\
 \therefore x &= 22 \cdot 2398.
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \text{Here } (x-1) \log 2 &= \log 5 = 1 - \log 2. \\
 \therefore x \log 2 &= 1; \\
 \therefore x &= \frac{1}{\log 2} = \frac{1}{\cdot 30103} = 3 \cdot 32.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \text{Here } (x-4) \log 3 &= \log 7. \\
 \therefore x-4 &= \frac{\log 7}{\log 3} = 1 \cdot 77; \\
 \therefore x &= 5 \cdot 77.
 \end{aligned}$$

11. Here $(1-x) \log 5 = (x-3) (\log 2 + \log 3)$;
 $\therefore x (\log 2 + \log 3 + \log 5) = 3 \log 2 + 3 \log 3 + \log 5$;
 $\therefore x = \frac{3 \cdot 0334239}{1 \cdot 4771213} = 2 \cdot 05.$

12. Put $\log 2 = a$, $\log 5 = b$; then
 $bx = -ay$, $b(2+y) = a(2-x).$

From these equations we obtain

$$x = \frac{2a}{a+b} = \frac{2 \log 2}{\log 2 + \log 5} = \frac{2 \log 2}{\log 10} = 2 \log 2 = \cdot 60206,$$

$$y = -\frac{ax}{b} = -2 \log 5 = -1 \cdot 39794.$$

13. Put $\log 2 = a$, $\log 3 = b$; then
 $ax = by$, $a(y+1) = b(x-1).$

From these equations we obtain

$$x = \frac{b}{b-a} = \frac{\log 3}{\log 3 - \log 2} = 2 \cdot 71,$$

$$y = \frac{a}{b-a} = \frac{\log 2}{\log 3 - \log 2} = x - 1 = 1 \cdot 71.$$

14. We have $2 \log 2 + \log 7 = a$, $\log 3 + \log 7 = b$, $2 - 2 \log 2 = c$.
 $\therefore 2 + \log 7 = a + c$, so that $\log 3 = b - a - c + 2$.
 $\therefore \log 27 = 3(b - a - c + 2).$

Again $\log 224 = \log (2^5 \times 7) = 5 \log 2 + \log 7$.
 $= \frac{5}{2} (2 - c) + (a + c - 2) = \frac{1}{2} (2a - 3c + 6).$

15. We have $\log (2 \times 11^2) = a$, $\log (2^3 \times 10) = b$, $\log (3^2 \times 5) = c$.
 $\therefore 2 \log 11 + \log 2 = a$, $3 \log 2 + 1 = b$, $2 \log 3 + 1 - \log 2 = c$.

From the last two equations

$$b + c - 2 = 2 \log 3 + 2 \log 2 = \log 36.$$

Again, $\log 66 = \log 6 + \log 11 = \frac{1}{2} (b + c - 2) + \frac{a - \log 2}{2}$
 $= \frac{1}{2} \left[b + c - 2 + a - \frac{1}{3} (b - 1) \right] = \frac{1}{6} (3a + 2b + 3c - 5).$

MISCELLANEOUS EXAMPLES. E. PAGE 150.

1. First side $= \frac{1}{2} (\cos 60^\circ + \cos 2A - \cos 120^\circ - \cos 2A)$
 $= \frac{1}{2} (\cos 60^\circ - \cos 120^\circ) = \frac{1}{2}.$

2. See Ex. 1, p. 118, and Ex. 3, p. 119.

$$3. \quad b^2 = a^2 + c^2 - 2ac \cos B = 4 + 2 - 2 \cdot 2 \cdot \sqrt{2} \cdot \frac{\sqrt{3}+1}{2\sqrt{2}} = 4 - 2\sqrt{3} + (\sqrt{3}-1)^2.$$

$$\therefore b = \sqrt{3} - 1.$$

$$\sin A = \frac{a}{b} \sin B = \frac{2}{\sqrt{3}-1} \cdot \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{1}{\sqrt{2}};$$

$\therefore A = 45^\circ$, or 135° ; but a is the greatest side, so that $A = 135^\circ$, and $C = 30^\circ$.

4. Each side easily reduces to 1.

$$5. \quad \text{Here } \frac{a}{b} = \frac{\cos A}{\cos B}; \text{ but in any triangle } \frac{a}{b} = \frac{\sin A}{\sin B}.$$

$$\therefore \frac{\sin A}{\sin B} = \frac{\cos A}{\cos B}, \text{ or } \sin(A-B) = 0;$$

$$\therefore A = B, \text{ and the triangle is isosceles.}$$

$$6. \quad (1) \text{ First side} = \frac{1}{2} (\cos 2\theta - \cos 4\theta + \cos 4\theta - \cos 6\theta + \cos 6\theta - \cos 8\theta + \cos 8\theta - \cos 10\theta) \\ = \frac{1}{2} (\cos 2\theta - \cos 10\theta) = \sin 6\theta \sin 4\theta.$$

$$(2) \text{ First side} = \frac{2 \sin 2a \cos a + 2 \sin 6a \cos a}{2 \cos 2a \cos a + 2 \cos 6a \cos a} \\ = \frac{\sin 2a + \sin 6a}{\cos 2a + \cos 6a} = \frac{2 \sin 4a \cos 2a}{2 \cos 4a \cos 2a} = \tan 4a.$$

$$7. \quad \text{First side} = \frac{\cos 3a \cos a + \sin 3a \sin a}{\sin a \cos a} = \frac{2 \cos (3a - a)}{2 \sin a \cos a} \\ = \frac{2 \cos 2a}{\sin 2a} = 2 \cot 2a.$$

$$8. \quad c^2 = a^2 + b^2 - 2ab \cos C = a^2 + (4 - 2\sqrt{3})a^2 - a^2(\sqrt{3}-1)\sqrt{3} = (2 - \sqrt{3})a^2. \\ \therefore a^2 = (2 + \sqrt{3})c^2, \text{ and } \sin^2 A = (2 + \sqrt{3}) \sin^2 C;$$

$$\therefore \sin^2 A = \frac{2 + \sqrt{3}}{4} = \frac{4 + 2\sqrt{3}}{8};$$

$$\therefore \sin A = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Hence $A = 75^\circ$, or 105° , and the latter value must be taken, as $A = 75^\circ$ would make the triangle isosceles. Hence also $B = 45^\circ$.

$$9. \quad \tan 4a = \frac{2 \tan 2a}{1 - \tan^2 2a}; \text{ substitute } \tan 2a = \frac{2 \tan a}{1 - \tan^2 a}.$$

$$\begin{aligned}
 10. \quad (1) \quad \text{First side} &= a^2 (1 - 2 \sin^2 B) + b^2 (1 - 2 \sin^2 A) \\
 &= a^2 + b^2 - 2a^2 \sin^2 B - 2b^2 \sin^2 A = a^2 + b^2 - 4a^2 \sin^2 B \\
 &= a^2 + b^2 - 4a \sin B \cdot b \sin A = a^2 + b^2 - 4ab \sin A \sin B.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \text{First side} &= 2bc (1 + \cos A) + \dots + \dots \\
 &= (2bc + b^2 + c^2 - a^2) + \dots + \dots = (a + b + c)^2.
 \end{aligned}$$

11. From the equation $c^4 - 2c^2(a^2 + b^2) + (a^4 + b^4) = 0$ we have

$$\{c^2 - (a^2 + ab\sqrt{2} + b^2)\} \{c^2 - (a^2 - ab\sqrt{2} + b^2)\} = 0.$$

Equating the two factors separately to zero, we get

$$\frac{a^2 + b^2 - c^2}{2ab} = \frac{1}{\sqrt{2}} \text{ or } -\frac{1}{\sqrt{2}};$$

whence

$$C = 45^\circ \text{ or } 135^\circ.$$

$$12. \quad \text{We have} \quad 2 \cos \frac{3(A+B)}{2} \cos \frac{3(A-B)}{2} + \cos 3C = 1;$$

$$\therefore 2 \cos \left(270^\circ - \frac{3C}{2}\right) \cos \frac{3(A-B)}{2} + \cos 3C = 1;$$

$$\therefore -2 \sin \frac{3C}{2} \cos \frac{3(A-B)}{2} + 1 - 2 \sin^2 \frac{3C}{2} = 1;$$

or

$$-2 \sin \frac{3C}{2} \left[\cos \frac{3(A-B)}{2} + \sin \frac{3C}{2} \right] = 0;$$

$$2 \sin \frac{3C}{2} \left[\cos \frac{3(A-B)}{2} - \cos \frac{3(A+B)}{2} \right] = 0;$$

$$4 \sin \frac{3A}{2} \sin \frac{3B}{2} \sin \frac{3C}{2} = 0.$$

Since A, B, C are the angles of a triangle we must have one of the angles $\frac{3A}{2}, \frac{3B}{2},$ or $\frac{3C}{2}$ equal to 180° . That is, one of the angles of the triangle must be 120° .

EXAMPLES. XV. a. PAGE 155.

$$\begin{aligned}
 1. \quad \log 49517 &= 4.6947543 \\
 \log 49516 &= 4.6947456 \\
 \text{diff. for } 1 &= \frac{87}{34}
 \end{aligned}$$

$$\frac{348}{261}$$

$$\text{diff. for } .34 = \frac{29}{58}$$

$$\log 49516 = 4.694756$$

$$\log 49516.34 = 4.694786$$

$$\therefore \log 4951634 = 6.694786.$$

$$\begin{aligned}
 2. \quad \log 3.4714 &= .5405047 \\
 \log 3.4713 &= .5404921 \\
 \text{diff. for } .0001 &= \frac{126}{206}
 \end{aligned}$$

$$\frac{756}{252}$$

$$\text{diff. for } .0000026 = \frac{3}{276}$$

$$\log 3.4713 = .5404921$$

$$\log 3.4713026 = .5404924$$

$$\begin{array}{rcl}
 3. \log 28497 & = & 4.4547991 \\
 \log 28496 & = & 4.4547839 \\
 \text{diff. for } 1 & = & \begin{array}{r} 152 \\ \cdot 14 \\ \hline 6 \overline{08} \\ 15 \overline{2} \\ \hline 21 \overline{28} \end{array} \\
 \text{diff. for } \cdot 14 = & & 21 \overline{28} \\
 \log 28496 & = & 4.4547839 \\
 \log 28496 \cdot 14 = & & 4.4547860 \\
 \therefore \log 2849614 = & & 6.4547860.
 \end{array}$$

$$\begin{array}{rcl}
 5. \log 60814 & = & 4.7840036 \\
 & 6 & 43 \overline{2} \\
 & 5 & 3 \overline{60} \\
 \therefore \log 6081465 = & & 6.7840083
 \end{array}$$

$$\begin{array}{rcl}
 7. \log x = 2.8283676 \\
 \log 67354 = 2.8283634 \\
 \text{diff.} = 42, \\
 \text{and diff. for } 1 = 64; \\
 \therefore \text{prop}^l. \text{ increase} = \frac{42}{64} = \frac{21}{32} = \cdot 66; \\
 \therefore x = 673.5466.
 \end{array}$$

$$\begin{array}{rcl}
 9. \log x = 3.9184377 \\
 \log .0082877 = 3.9184340 \\
 \text{diff.} = 37, \\
 \text{and diff. for } 1 = 52; \\
 \therefore \text{prop}^l. \text{ increase} = \frac{37}{52} = \cdot 71; \\
 \therefore x = .008287771.
 \end{array}$$

$$\begin{array}{rcl}
 11. \log x = \frac{1}{7} \log 142.71 \\
 = \frac{1}{7} (2.1544544) \\
 = .3077792 \\
 \log 2.0313 = .3077741 \\
 \text{diff.} = 51, \\
 \text{and diff. for } 1 = 213; \\
 \therefore \text{prop}^l. \text{ increase} = \frac{51}{213} = \cdot 24; \\
 \therefore x = 2.031324.
 \end{array}$$

$$\begin{array}{rcl}
 4. \log 57.634 & = & 1.7606788 \\
 \log 57.633 & = & 1.7606712 \\
 \text{diff. for } .001 & = & \begin{array}{r} 76 \\ \cdot 25 \\ \hline 3 \overline{80} \\ 15 \overline{2} \\ \hline 19 \overline{00} \end{array} \\
 \text{diff. for } .00025 = & & 19 \overline{00} \\
 \log 57.633 & = & 1.7606712 \\
 \log 57.63325 = & & 1.7606731
 \end{array}$$

$$\begin{array}{rcl}
 6. \log x = 4.7461735 \\
 \log 55740 = 4.7461670 \\
 \text{diff.} = 65, \\
 \text{and diff. for } 1 = 78; \\
 \therefore \text{prop}^l. \text{ increase} = \frac{65}{78} = \frac{5}{6} = \cdot 83; \\
 \therefore x = 55740.83.
 \end{array}$$

$$\begin{array}{rcl}
 8. \log x = 2.0288435 \\
 \log .010686 = 2.0288152 \\
 \text{diff.} = 283, \\
 \text{and diff. for } 1 = 406; \\
 \therefore \text{prop}^l. \text{ increase} = \frac{283}{406} = \cdot 7; \\
 \therefore x = .0106867.
 \end{array}$$

$$\begin{array}{rcl}
 10. \log x = 1.4034508 \\
 \log .25319 = 1.4034465 \\
 \text{diff.} = 43, \\
 \therefore \text{prop}^l. \text{ increase} = \frac{43}{172} = \frac{1}{4} = \cdot 25; \\
 \therefore x = .2531925.
 \end{array}$$

$$\begin{array}{rcl}
 12. \log 13.894 & = & 1.1428273 \\
 & 9 & 281 \overline{7} \\
 & 2 & 6 \overline{26} \\
 & 8 & 1.1428561 \\
 \log x & = & .1428570 \\
 \log 1.3894 = & & .1428273 \\
 \text{diff.} & = & 297 \\
 \text{and diff. for } 1 = 313; \\
 \therefore \text{prop}^l. \text{ increase} = \frac{297}{313} = \cdot 95; \\
 \therefore x = 1.389495.
 \end{array}$$

$$\begin{array}{rcl}
 13. \log 24244 & = & 5.3846043 \\
 & & 7 \quad 125 \overline{)3} \\
 & & 14 \overline{)5.3846168} \\
 \log x & = & .3846155 \\
 \log 2.4244 & = & .3846043 \\
 \text{diff.} & = & \overline{112} \\
 \therefore \text{prop}^l. \text{ increase} & = & \frac{112}{179} = .63; \\
 \therefore x & = & 2.424463.
 \end{array}$$

$$\begin{array}{rcl}
 14. \log 20691 & = & 6.3157815 \\
 & & 3 \quad 63 \overline{)0} \\
 & & 8 \quad 16 \overline{)80} \\
 & & 20 \overline{)6.3157895} \\
 \log x & = & .3157895 \\
 \log 2.0691 & = & .3157815 \\
 \text{diff.} & = & \overline{80} \\
 \therefore \text{prop}^l. \text{ increase} & = & \frac{80}{210} = \frac{8}{21} = .38; \\
 \therefore x & = & 2.069138.
 \end{array}$$

EXAMPLES. XV. b. PAGE 159.

1. $\sin 38^\circ 3' = .6163489$; diff. for $60'' = 2291$.
 $\text{prop}^l. \text{ increase} = \frac{35}{60} \times 2291 = \frac{1336}{.6164825}$
3. $\text{cosec } 55^\circ 21' = 1.2155978$; diff. for $60'' = 2443$.
 $\text{prop}^l. \text{ decrease} = \frac{28}{60} \times 2443 = \frac{1140}{1.2154838}$
4. $\sec \theta - \sec 62^\circ 42' = 6321$;
 diff. for $60'' = 12296$;
 and $\frac{6321}{12296} \times 60'' = 31''$;
 $\therefore \theta = 62^\circ 42' 31''$.
5. $\cos 30^\circ 40' - \cos \theta = 560$;
 diff. for $60'' = 1484$;
 and $\frac{560}{1484} \times 60'' = 23''$;
 $\therefore \theta = 30^\circ 40' 23''$.
6. $\cot 48^\circ 45' - \cot \theta = 3762$;
 diff. for $60'' = 5145$;
 and $\frac{3762}{5145} \times 60'' = 44''$;
 $\therefore \theta = 48^\circ 45' 44''$.
7. $L \sin 44^\circ 17' = 9.8439842$; diff. for $60'' = 1295$.
 $\text{prop}^l. \text{ increase} = \frac{33}{60} \times 1295 = \frac{712}{9.8440554}$
9. $L \cos 55^\circ 30' = 9.7531280$; diff. for $60'' = 1838$.
 $\text{prop}^l. \text{ decrease} = \frac{24}{60} \times 1838 = \frac{735}{9.7530545}$

10. $L \sin \theta - L \sin 44^\circ 17' = 176$; 11. $L \cos 55^\circ 30' - L \cos \theta = 1205$;
 diff. for $60'' = 1295$; diff. for $60'' = 1838$;
 and $\frac{176}{1295} \times 60'' = 8''$; and $\frac{1205}{1838} \times 60'' = 39''$;
 $\therefore \theta = 44^\circ 17' 8''$. $\therefore \theta = 55^\circ 30' 39''$.

12. $L \tan 24^\circ 50' = 9.6653662$; diff. for $60'' = 3313$.
 prop^l. increase = $\frac{52.5}{60} \times 3313 = \frac{2899}{9.6656561}$

13. The required angle is $42.5''$ less than $40^\circ 5'$;
 $\therefore$ prop^l. increase = $\frac{42.5}{60} \times 1502 = 1064$
 $L \operatorname{cosec} 40^\circ 5' = 10.1911808$
 $L \operatorname{cosec} 40^\circ 4' 17.5'' = 10.1912872$

EXAMPLES. XV. c. PAGE 161.

1. $\log 300.26 = 2.4774975$
 1 15
 8 11 6
 $\log .0078915 = 3.8971596$
 1 6
 9 5 0
 4 22
 3746609
 $\log 2.3695 = 3.746567$
 42
 2 37

Thus the product = 2.36952.

2. $\log 235.67 = 2.3723043$
 8 148
 3 5 6
 $\log 357.84 = 2.5536889$
 3 36
 8 9 8
 4.9260131
 $\log 84336 = 4.9260130$

Thus the product is 84336.

3.

log 153·24	= 2·1853721	
		1 28
		9 25 6
log 2·8632	= ·4568517	
		5 76
		0 0
		3 46
log ·075836	= 2·8798754	
		4 23
		6 3 4
	1·5221148	
log 33·274	= 1·5221050	
		7 98
		5 91
		70
		65

Thus the product is 33·27475.

4. log 1·0304 = ·0130059 <table style="margin-left: 100px;"> <tr><td></td><td style="text-align: right;">0</td><td style="border-left: 1px solid black; padding-left: 5px;">0</td></tr> <tr><td></td><td style="text-align: right;">5</td><td style="border-left: 1px solid black; padding-left: 5px;">21 1</td></tr> <tr><td></td><td style="text-align: right;">1</td><td style="border-left: 1px solid black; padding-left: 5px;">42</td></tr> <tr><td colspan="3"> </td></tr> <tr><td></td><td></td><td style="border-left: 1px solid black; padding-left: 5px;">·0130081</td></tr> <tr><td style="text-align: right;">subtract</td><td></td><td style="border-left: 1px solid black; padding-left: 5px;">1·4328656</td></tr> <tr><td colspan="3"> </td></tr> <tr><td></td><td></td><td style="border-left: 1px solid black; padding-left: 5px;">2·5801425</td></tr> <tr><td style="text-align: right;">log ·038031</td><td></td><td style="border-left: 1px solid black; padding-left: 5px;">= 2·5801377</td></tr> <tr><td colspan="3"> </td></tr> <tr><td></td><td></td><td style="border-left: 1px solid black; padding-left: 5px;">48</td></tr> <tr><td></td><td style="text-align: right;">4</td><td style="border-left: 1px solid black; padding-left: 5px;">46</td></tr> <tr><td colspan="3"> </td></tr> <tr><td></td><td></td><td style="border-left: 1px solid black; padding-left: 5px;">20</td></tr> <tr><td></td><td style="text-align: right;">2</td><td style="border-left: 1px solid black; padding-left: 5px;">23</td></tr> <tr><td colspan="3"> </td></tr> </table>		0	0		5	21 1		1	42						·0130081	subtract		1·4328656						2·5801425	log ·038031		= 2·5801377						48		4	46						20		2	23				log 27·093 = 1·4328571 <table style="margin-left: 100px;"> <tr><td></td><td style="text-align: right;">5</td><td style="border-left: 1px solid black; padding-left: 5px;">81</td></tr> <tr><td></td><td style="text-align: right;">2</td><td style="border-left: 1px solid black; padding-left: 5px;">3 2</td></tr> <tr><td></td><td style="text-align: right;">4</td><td style="border-left: 1px solid black; padding-left: 5px;">64</td></tr> <tr><td colspan="3"> </td></tr> <tr><td></td><td></td><td style="border-left: 1px solid black; padding-left: 5px;">1·4328656</td></tr> </table>		5	81		2	3 2		4	64						1·4328656
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Thus the quotient is ·03803142.

5. log 357·83 = 2·5536767 <table style="margin-left: 100px;"> <tr><td></td><td style="text-align: right;">6</td><td style="border-left: 1px solid black; padding-left: 5px;">73</td></tr> <tr><td></td><td style="text-align: right;">4</td><td style="border-left: 1px solid black; padding-left: 5px;">4 8</td></tr> <tr><td colspan="3"> </td></tr> <tr><td></td><td></td><td style="border-left: 1px solid black; padding-left: 5px;">2·5536845</td></tr> <tr><td></td><td></td><td style="border-left: 1px solid black; padding-left: 5px;">3·5037539</td></tr> <tr><td colspan="3"> </td></tr> <tr><td></td><td></td><td style="border-left: 1px solid black; padding-left: 5px;">5·0499306</td></tr> <tr><td style="text-align: right;">log 11218</td><td></td><td style="border-left: 1px solid black; padding-left: 5px;">= 4·0499154</td></tr> <tr><td colspan="3"> </td></tr> <tr><td></td><td></td><td style="border-left: 1px solid black; padding-left: 5px;">152</td></tr> <tr><td></td><td style="text-align: right;">4</td><td style="border-left: 1px solid black; padding-left: 5px;">155</td></tr> <tr><td colspan="3"> </td></tr> </table>		6	73		4	4 8						2·5536845			3·5037539						5·0499306	log 11218		= 4·0499154						152		4	155				log ·0031897 = 3·5037498 <table style="margin-left: 100px;"> <tr><td></td><td style="text-align: right;">3</td><td style="border-left: 1px solid black; padding-left: 5px;">41</td></tr> <tr><td colspan="3"> </td></tr> <tr><td></td><td></td><td style="border-left: 1px solid black; padding-left: 5px;">3·5037539</td></tr> </table>		3	41						3·5037539
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Thus the quotient is 112184.

6. $\log 21.856 = 1.3395707$ $\begin{array}{r} 3 \qquad 60 \\ 2 \qquad 40 \end{array}$ 1.3395771 subtract $\underline{2.2512610}$ $\log x = 3.0883161$ $\log 1225.5 = 3.0883133$ $\begin{array}{r} 0 \qquad 280 \\ 8 \qquad 284 \end{array}$ $\therefore x = 1225.508.$	$\log .017834 = \bar{2}.2512488$ $\begin{array}{r} 5 \qquad 122 \end{array}$ $\underline{2.2512610}$
--	--

7. $\log 3.7895 = .5785819$

$\begin{array}{r} 6 \qquad 69 \end{array}$

$\log .053687 = \bar{2}.7298691$

$\begin{array}{r} 2 \qquad 16 \end{array}$

1.3084595

$\log .0072916 = \bar{3}.8628228$

1.4456367

$\log 27.902 = 1.4456353$

$\begin{array}{r} 0 \qquad 140 \\ 9 \qquad 140 \end{array}$

Thus the required value is 27.90209.

8. $\log .83410 = 1.9212181$

$\begin{array}{r} 0 \qquad 0 \\ 3 \qquad 16 \\ 9 \qquad 47 \end{array}$

1.9212183

$\begin{array}{r} 3 \end{array}$

1.7636549

$\log .58030 = 1.7636526$

$\begin{array}{r} 23 \\ 3 \qquad 22 \end{array}$

Thus the cube is .580303.

9. $\log 15063 = 4.1779115$

$\begin{array}{r} 0 \qquad 0 \\ 1 \qquad 29 \\ 8 \qquad 230 \end{array}$

5 | 4.1779120

.8355824

$\log 6.8482 = .8355764$

$\begin{array}{r} 60 \\ 9 \qquad 58 \end{array}$

$\begin{array}{r} 20 \\ 3 \qquad 19 \end{array}$

Thus the fifth root is 6.848293.

$$\begin{array}{rcl}
 10. \log 384.73 & = & 2.5851561 \\
 & 1 & 11 \\
 & 5 & \overline{2.5851572} \\
 & & .5170314 \\
 \log 3.2887 & = & .5170243 \\
 & & \overline{71} \\
 & 5 & 66 \\
 & & \overline{50} \\
 & 4 & 53
 \end{array}$$

Thus $\sqrt[5]{384.73} = 3.288754$.

$$\begin{array}{rcl}
 \log 15.732 & = & 1.1967839 \\
 & 4 & 111 \\
 & 13 & \overline{1.1967950} \\
 & & .0920612 \\
 \log 1.2361 & & .0920536 \\
 & & \overline{76} \\
 & 2 & 70 \\
 & & \overline{60} \\
 & 2 & 70
 \end{array}$$

Thus $\sqrt[13]{15.7324} = 1.236122$.

$$\begin{array}{rcl}
 11. \log 1034.3 & = & 3.0146465 \\
 & 9 & 379 \\
 & 6 & 253 \\
 & 3 & 126 \\
 & 2 & \overline{3.0146871} \\
 & & 1.5073435 \\
 \text{add} & & 1.8493591 \\
 & & 3.3567026 \\
 \log 2273.5 & = & \overline{3.3566950} \\
 & & 76 \\
 & 4 & 76
 \end{array}$$

$$\begin{array}{rcl}
 \log 35324 & = & 5.5480699 \\
 & 6 & 74 \\
 & 3 & \overline{5.5480773} \\
 & & 1.8493591
 \end{array}$$

Thus the product is 2273.54.

12. Let $a = 1.0356270$ and $b = .7503269$; then $a^2 - b^2 = (a + b)(a - b)$, and $a + b = 1.7859539$, $a - b = .2853001$.

$$\begin{array}{rcl}
 \log 1.7859 & = & .2518571 \\
 & 5 & 122 \\
 & 3 & 73 \\
 & 9 & 219 \\
 \log .28530 & = & \overline{1.4553018} \\
 & 0 & 0 \\
 & 1 & 15 \\
 & & \overline{1.7071722} \\
 \log .50953 & & \overline{1.7071698} \\
 & & 24 \\
 & 2 & 17 \\
 & & \overline{70} \\
 & 8 & 69
 \end{array}$$

Thus the difference is .5095328.

$$13. \log x = \frac{3}{5} \log 34.7326 + \frac{1}{6} \log 2.53894 - \frac{1}{5} \log 4.39682.$$

$$\begin{array}{r} \log 2.5389 \quad = \cdot 4046456 \\ \quad \quad \quad 4 \quad \quad \quad 68 \\ \quad \quad \quad 6 \overline{) \cdot 4046524} \\ \quad \quad \quad \cdot 0674421 \\ \text{add} \quad \quad \quad \cdot 7958146 \\ \quad \quad \quad \cdot 8632567 \\ \log 7.2988 \quad = \cdot 8632515 \\ \quad \quad \quad \quad \quad \quad 52 \\ \quad \quad \quad 9 \quad \quad \quad 54 \end{array}$$

$$\begin{array}{r} \log 4.3968 \quad = \cdot 6431367 \\ \quad \quad \quad 2 \quad \quad \quad 20 \\ \quad \quad \quad \cdot 6431387 \\ \log 34.732 \quad = 1.5407298 \\ \quad \quad \quad 6 \quad \quad \quad 75 \\ \quad \quad \quad 1.5407373 \\ \quad \quad \quad \quad \quad 3 \\ \quad \quad \quad 4.6222119 \\ \text{subtract} \quad \quad \quad \cdot 6431387 \\ \quad \quad \quad 5 \overline{) 3.9790732} \\ \quad \quad \quad \cdot 7958146 \end{array}$$

Thus $x = 7.29889$.

$$\begin{array}{r} 14. \log .0037258 \quad = \bar{3}.5712195 \\ \quad \quad \quad 1 \quad \quad \quad 12 \\ \quad \quad \quad 6 \quad \quad \quad 70 \\ \quad \quad \quad 9 \quad \quad \quad 105 \\ \quad \quad \quad \bar{3}.5712215 \\ \text{add} \quad \quad \quad \bar{1}.7505167 \\ \quad \quad \quad 2 \overline{) \bar{3}.3217382} \\ \quad \quad \quad \bar{2}.6608691 \\ \log .045800 \quad = \bar{2}.6608655 \\ \quad \quad \quad \quad \quad \quad 36 \\ \quad \quad \quad 3 \quad \quad \quad 29 \\ \quad \quad \quad \quad \quad \quad 70 \\ \quad \quad \quad 7 \quad \quad \quad 67 \end{array}$$

$$\begin{array}{r} \log .56301 \quad = \bar{1}.7505161 \\ \quad \quad \quad 0 \quad \quad \quad 0 \\ \quad \quad \quad 7 \quad \quad \quad 54 \\ \quad \quad \quad 8 \quad \quad \quad 62 \\ \quad \quad \quad \bar{1}.7505167 \end{array}$$

Thus the mean proportional is $.04580037$.

$$15. \text{ If } x \text{ be the required number, we have } x = \frac{\cdot 03751786}{(\cdot 43607528)^2},$$

$$\begin{array}{r} \log .037517 \quad = \bar{2}.5742281 \\ \quad \quad \quad 8 \quad \quad \quad 93 \\ \quad \quad \quad 6 \quad \quad \quad 70 \\ \quad \quad \quad \bar{2}.5742381 \\ \quad \quad \quad \bar{1}.2791230 \\ \quad \quad \quad \bar{1}.2951151 \\ \log .19729 \quad \bar{1}.2951051 \\ \quad \quad \quad \quad \quad \quad 100 \\ \quad \quad \quad 4 \quad \quad \quad 88 \\ \quad \quad \quad \quad \quad \quad 120 \\ \quad \quad \quad 5 \quad \quad \quad 110 \end{array}$$

$$\begin{array}{r} \log .43607 \quad = \bar{1}.6395562 \\ \quad \quad \quad 5 \quad \quad \quad 50 \\ \quad \quad \quad 2 \quad \quad \quad 20 \\ \quad \quad \quad 8 \quad \quad \quad 79 \\ \quad \quad \quad \bar{1}.6395615 \\ \quad \quad \quad \quad \quad 2 \\ \quad \quad \quad \bar{1}.2791230 \end{array}$$

Thus $x = .1972945$.

16. If x be the required number, we have

$$x = \frac{29 \cdot 302564 \times \cdot 33025107}{56712 \cdot 43}.$$

$\begin{array}{r} \log 29 \cdot 302 \quad = 1 \cdot 4668973 \\ \quad \quad \quad 5 \quad \quad \quad 74 \\ \quad \quad \quad 6 \quad \quad \quad 89 \\ \quad \quad \quad 4 \quad \quad \quad 59 \\ \log \cdot 33025 \quad = 1 \cdot 5188428 \\ \quad \quad \quad 1 \quad \quad \quad 13 \\ \quad \quad \quad 0 \quad \quad \quad 0 \\ \quad \quad \quad 7 \quad \quad \quad 92 \\ \hline \quad \quad \quad \cdot 9857498 \\ \text{subtract} \quad \quad 4 \cdot 7536783 \\ \hline \quad \quad \quad 4 \cdot 2320715 \\ \log \cdot 00017063 \quad = 4 \cdot 2320554 \\ \hline \quad \quad \quad 161 \\ \quad \quad \quad 6 \quad \quad \quad 152 \\ \hline \quad \quad \quad 90 \\ \quad \quad \quad 3 \quad \quad \quad 76 \\ \hline \end{array}$	$\begin{array}{r} \log 56712 \quad = 4 \cdot 7536750 \\ \quad \quad \quad 4 \quad \quad \quad 31 \\ \quad \quad \quad 3 \quad \quad \quad 23 \\ \hline \quad \quad \quad 4 \cdot 7536783 \end{array}$
---	---

Thus the fourth proportional is $\cdot 0001706363$.

17. Let x be the required number, then

$$x = \sqrt{(\cdot 035689)^{\frac{2}{5}} \times (2 \cdot 879432)^{\frac{3}{7}}}.$$

$\begin{array}{r} \log 2 \cdot 8794 \quad = \quad \cdot 4593020 \\ \quad \quad \quad 3 \quad \quad \quad 45 \\ \quad \quad \quad 2 \quad \quad \quad 30 \\ \hline \quad \quad \quad \cdot 4593068 \\ \quad \quad \quad \quad \quad 3 \\ 14 \quad \hline 1 \cdot 3779204 \\ \quad \quad \quad \cdot 0984229 \\ \quad \quad \quad 1 \cdot 7105069 \\ \hline \quad \quad \quad 1 \cdot 8089298 \\ \log \cdot 64406 \quad = \quad 1 \cdot 8089263 \\ \hline \quad \quad \quad 35 \\ \quad \quad \quad 5 \quad \quad \quad 34 \\ \hline \end{array}$	$\begin{array}{r} \frac{1}{5} \log \cdot 035689 = \frac{1}{5} (\bar{2} \cdot 5525344) \\ \hline = \bar{1} \cdot 7105069 \end{array}$
---	--

Thus the geometric mean is $\cdot 644065$.

18. Here $x = \frac{(7836 \cdot 43)^{\frac{1}{4}} \times (357 \cdot 814)^{\frac{1}{5}}}{(32 \cdot 7812)^{\frac{1}{3}}}$.

log 7836·4	= 3·8941166
3	17
	4 3·8941183
	·9735295
add	·5107315
	1·4842610
subtract	·5052083
	·9790527
log 9·5291	= ·9790519
	8
2	9

log 32·781	= 1·5156222
2	26
	3 1·5156248
	·5052083
log 357·81	= 2·5536525
4	49
	5 2·5536574
	·5107315

Thus the fourth proportional is 9·52912.

19. $\log \sin 27^\circ 13' = \bar{1} \cdot 6602550$

$\frac{12}{60} \times 2455 =$	491
	1·6603041
	1·8414768
	1·5017809
log ·31752	= 1·5017711
	98
7	96
	20
1	14

$\log \cos 46^\circ 2' = \bar{1} \cdot 8415095$

subtract $\frac{15}{60} \times 1310 =$	327
	1·8414768

Thus the required value is ·3175271.

20. $\cot 97^\circ 14' 16'' = -\cot 82^\circ 45' 44''$,
 $\sec 112^\circ 13' 5'' = -\sec 67^\circ 46' 55''$.

log sec 67° 46'	= 4220725
$\frac{11}{12} \times 3092 =$	2834
	4223559
	1·1038011
	1·5261570
log ·33585	= 1·5261454
	116
9	116

log cot 82° 45'	= 1·1045420
$\frac{11}{15} \times 10103 =$	7409
	1·1038011

Thus the required value is ·335859.

$$\begin{array}{rcl}
 21. \quad \log \sin 20^\circ 13' & = & \bar{1} \cdot 5385375 \\
 \frac{20}{60} \times 3429 & = & 1143 \\
 \log \sec 42^\circ 15' & = & \cdot 1306403 \\
 \frac{30}{60} \times 1148 & = & 574 \\
 & & \hline
 & & 1 \cdot 6693495
 \end{array}$$

$$\begin{array}{rcl}
 \log \cot 47^\circ 53' & = & \bar{1} \cdot 9562154 \\
 \text{subtract } \frac{15}{60} \times 2540 & = & 635 \\
 & & \hline
 & & \bar{1} \cdot 9561519 \\
 \text{add} & & \bar{1} \cdot 6693495 \\
 & & \hline
 & & \bar{1} \cdot 6255014 \\
 \log \cdot 42218 & = & \bar{1} \cdot 6254977 \\
 & & \hline
 & & 37 \\
 & & 3 \quad 31 \\
 & & \hline
 & & 60 \\
 & & 6 \quad 62 \\
 & & \hline
 & & 62
 \end{array}$$

Thus the required value is $\cdot 4221836$.

$$\begin{array}{rcl}
 22. \quad \log 324 \cdot 13 & = & 2 \cdot 5107192 \\
 & 6 & 80 \\
 & 8 & 10 \quad 7 \\
 \log 417 \cdot 24 & = & 2 \cdot 6203859 \\
 & 3 & 31 \\
 & 1 & 10 \\
 & & \hline
 & & 5 \cdot 1311174 \\
 & & \bar{1} \cdot 9632566 \\
 & & \hline
 & & 5 \cdot 0943740 \\
 \log 12427 & = & 5 \cdot 0943663 \\
 & & \hline
 & & 77 \\
 & 2 & 70 \\
 & & \hline
 & & 70 \\
 & 2 & 70 \\
 & & \hline
 & & 70
 \end{array}$$

$$\begin{array}{rcl}
 \log \sin 113^\circ 14' 16'' & & \\
 = \log \sin 66^\circ 45' 44'', & & \\
 \log \sin 66^\circ 45' & = & \bar{1} \cdot 9632168 \\
 \frac{44}{60} \times 543 & = & 398 \\
 & & \hline
 & & \bar{1} \cdot 9632566
 \end{array}$$

Thus the required value is $12427 \cdot 2$.

$$23. \quad \text{Here } a = \frac{b \sin A}{\sin B}, \text{ and } \sin B = \sin 60^\circ 45' 42''.$$

$$\begin{array}{rcl}
 \log \sin 35^\circ 15' & = & \bar{1} \cdot 7612851 \\
 \frac{33}{60} \times 1787 & = & 983 \\
 & & \hline
 & & \bar{1} \cdot 7613834 \\
 & & \hline
 & & \bar{1} \cdot 9408130 \\
 & & \hline
 & & \bar{1} \cdot 8205704 \\
 \log 378 \cdot 25 & = & 2 \cdot 5777789 \\
 & & \hline
 & & 2 \cdot 3983493 \\
 \log 250 \cdot 23 & = & 2 \cdot 3983394 \\
 & & \hline
 & & 99 \\
 & 5 & 87 \\
 & & \hline
 & & 120 \\
 & 7 & 121 \\
 & & \hline
 & & 121
 \end{array}$$

$$\begin{array}{rcl}
 \log \sin 60^\circ 45' & = & \bar{1} \cdot 9407634 \\
 \frac{42}{60} \times 708 & = & 496 \\
 & & \hline
 & & \bar{1} \cdot 9408130
 \end{array}$$

Thus $a = 250 \cdot 2357$.

24. (1) $\log \tan \theta = \frac{1}{3} (\log 5 - \log 12)$
 $\log 5 = \cdot 6989700$
 $\log 12 = 1\cdot 0791812$
 $3 \overline{1\cdot 6197888}$
 $\log \tan \theta = 1\cdot 8732629$
 $\log \tan 36^\circ 45' = 1\cdot 8731668$
 $\qquad\qquad\qquad 961$
 $\frac{961}{2634} \times 60'' = 22''.$
 $\therefore \theta = 36^\circ 45' 22''.$

(2) $(3 \sin \theta - 1) (\sin \theta + 1) = 0.$
 $\therefore \sin \theta = \frac{1}{3} \text{ or } -1.$
 $\log \sin \theta = -\log 3$
 $\qquad\qquad\qquad = 1\cdot 5228787$
 $\log \sin 19^\circ 28' = 1\cdot 5227811$
 $\qquad\qquad\qquad\qquad\qquad\qquad 976$
 $\frac{976}{3572} \times 60'' = 16''.$
 $\therefore \theta = 19^\circ 28' 16''.$

25. $x = \sin 23^\circ 18' 5'' \times \cot 38^\circ 15' 13'' \times \cos 28^\circ 17' 25''.$

$\log \cot 38^\circ 15' = \cdot 1032884$
 $subtract \frac{13}{60} \times 2598 = \underline{\qquad 562 \qquad}$
 $\qquad\qquad\qquad \cdot 1032322$
 $\qquad\qquad\qquad \overline{1\cdot 9447579}$
 $\log \sin 23^\circ 18' = 1\cdot 5971965$
 $\frac{5}{60} \times 2932 = \underline{\qquad 244 \qquad}$
 $\qquad\qquad\qquad \overline{1\cdot 6452110}$
 $\log \cdot 44178 = \overline{1\cdot 6452061}$
 $\qquad\qquad\qquad\qquad\qquad\qquad 49$
 $\qquad\qquad\qquad 5 \qquad\qquad\qquad 49$

$\log \cos 28^\circ 17' = 1\cdot 9447862$
 $subtract \frac{25}{60} \times 680 = \underline{\qquad 283 \qquad}$
 $\qquad\qquad\qquad \overline{1\cdot 9447579}$

Thus $x = \cdot 441785.$

26. $\log \cos 32^\circ 47' = \overline{1\cdot 9246535}$
 $\qquad\qquad\qquad\qquad\qquad\qquad 2$
 $\qquad\qquad\qquad\qquad\qquad\qquad \overline{1\cdot 8493070}$
 $\log \cot 41^\circ 19' = \cdot 0559928$
 $\qquad\qquad\qquad\qquad\qquad\qquad 3 \overline{1\cdot 9052998}$
 $\qquad\qquad\qquad\qquad\qquad\qquad \overline{1\cdot 9684333}$
 $\log \sin 68^\circ 25' = \overline{1\cdot 9684286}$
 $\qquad\qquad\qquad\qquad\qquad\qquad 47$
 $\frac{47}{499} \times 60'' = 6''.$

Thus $\theta = 68^\circ 25' 6''.$

EXAMPLES. XV. d. PAGE 163 C.

- | | |
|--|--|
| <p>1. $\log 2834 = 3.4524$
 $\log 17.62 = 1.2460$
 $\log x = \overline{4.6984}$;
 whence $x = 49940$.</p> | <p>2. $\log 8.034 = .9049$
 $\log 1893 = 3.2772$
 $\log x = \overline{4.1821}$;
 whence $x = 15210$.</p> |
| <p>3. $\log .00567 = \overline{3.7536}$
 $\log .0297 = \overline{2.4728}$
 $\log x = \overline{4.2264}$;
 whence $x = .0001685$.</p> | <p>4. $\log 3.7 = .5682$
 $\log 8.9 = .9494$
 $\log .023 = \overline{2.3617}$
 $\log x = \overline{1.8793}$;
 whence $x = .7573$.</p> |
| <p>5. $\log 31.9 = 1.5038$
 $\log 1.51 = .1790$
 $\log 9.7 = .9868$
 $\log x = \overline{2.6696}$;
 whence $x = 467.3$.</p> | <p>6. $\log 43 = 1.6335$
 $\log 8.07 = .9069$
 $\log .0392 = \overline{2.5933}$
 $\log x = \overline{1.1337}$;
 whence $x = 13.60$.</p> |
| <p>7. $\log 17.3 = 1.2380$
 $\log 294.8 = 2.4695$
 $\log x = \overline{2.7685}$;
 whence $x = .05868$.</p> | <p>8. $\log 2.035 = .3086$
 $\log 837.6 = 2.9230$
 $\log x = \overline{3.3856}$;
 whence $x = .00243$.</p> |
| <p>9. $\log .2179 = \overline{1.3383}$
 $\log .08973 = \overline{2.9529}$
 $\log x = .3854$;
 whence $x = 2.429$.</p> | <p>10. $\log 487 = 2.6875$
 $\log 6398 = 3.8060$
 $\log x = \overline{2.8815}$;
 whence $x = .07612$.</p> |
| <p>11. $\log 2.38 = .3766$
 $\log 3.901 = .5912$
 $\log 4.83 = \overline{.9678}$
 $\log x = .6839$
 $\log x = .2839$;
 whence $x = 1.923$.</p> | <p>12. $\log 14.72 = 1.1679$
 $\log 38.05 = 1.5804$
 $\log 387.9 = \overline{2.7483}$
 $\log x = 2.5887$
 $\log x = .1596$;
 whence $x = 1.444$.</p> |
| <p>13. $\log 925.9 = 2.9665$
 $\log 1.597 = .2034$
 $\log 74.03 = \overline{3.1699}$
 $\log x = 1.8694$
 $\log x = \overline{1.3005}$;
 whence $x = 19.97$.</p> | |
| <p>14. $\log 15.38 = 1.1869$
 $\log .0137 = \overline{2.1367}$
 $\log x = \overline{1.3236}$
 $\log x = .0207$
 $\log x = \overline{1.3029}$;
 whence $x = .2008$.</p> | <p>$\log 276 = 2.4409$
 $\log .0038 = \overline{3.5798}$
 $\log x = .0207$</p> |

$$\begin{array}{ll}
 15. & \log 2.31 = \cdot 3636 & \log .0561 = \bar{2}.7490 \\
 & \log .037 = \bar{2}.5682 & \log 3.87 = \cdot 5877 \\
 & \log 1.43 = \cdot 1553 & \log .0091 = \bar{3}.9590 \\
 & \quad \bar{1}.0871 & \quad \bar{3}.2957 \\
 & \quad \bar{3}.2957 &
 \end{array}$$

$$\log x = \bar{1}.7914; \text{ whence } x = 61.86.$$

$$16. \log x = \frac{1}{2} \log 5.1 = \frac{1}{2} (\cdot 7076) = \cdot 3538; \text{ whence } x = 2.258.$$

$$17. \log x = \frac{1}{3} \log 11 = \frac{1}{3} (1.0414) = \cdot 3471; \text{ whence } x = 2.224.$$

$$18. \log x = \frac{1}{3} \log 82.56 = \frac{1}{3} (1.9168) = \cdot 6389; \text{ whence } x = 4.354.$$

$$19. \log x = \frac{1}{4} \log 10.15 = \frac{1}{4} (1.0064) = \cdot 2516; \text{ whence } x = 1.784.$$

$$20. \log x = 4 \log .097 = 4 (\bar{2}.9868) = \bar{5}.9472; \text{ whence } x = .00008855.$$

$$21. 5 \log 2.301 = \cdot 3619 \times 5 = 1.8095; \text{ whence } x = 64.49.$$

$$22. \frac{2}{3} \log 51.32 = \frac{1.7103 \times 2}{3} = 1.1402; \text{ whence } x = 13.81.$$

$$23. \frac{4}{7} \log .089 = \frac{\bar{2}.9494 \times 4}{7} = \bar{1}.3997; \text{ whence } x = .2510.$$

$$24. \log .0137 = \bar{2}.1367$$

$$\log .0296 = \bar{2}.4713$$

$$= 4.6080$$

$$\log 873.5 = 2.9412$$

$$2) \bar{7}.6668$$

$$\bar{4}.8334; \text{ whence } x = .0006814.$$

$$25. \log 83 = 1.9191$$

$$\frac{1}{3} \log 92 = \cdot 6546$$

$$2.5737$$

$$2.5820$$

$$\log x = \bar{1}.9917; \text{ whence } x = .9811.$$

$$\log 127 = 2.1038$$

$$\frac{1}{5} \log 246 = \cdot 4782$$

$$2.5820$$

$$26. \log .678 = \bar{1}.8312$$

$$\log 9.01 = \cdot 9547$$

$$\cdot 7859$$

$$\log .0234 = \bar{2}.3692$$

$$2) 2.4167$$

$$\log x = 1.2084 = \log 16.15;$$

$$\therefore x = 16, \text{ to nearest integer.}$$

27. (i) If x is the mean proportional between 2.87 and 30.08,

$$\begin{aligned} x &= \sqrt{2.87 \times 30.08} \\ \log 2.87 &= .4579 \\ \log 30.08 &= 1.4782 \\ 2) \overline{1.9361} \\ \log x &= .9680; \text{ whence } x = 9.29. \end{aligned}$$

- (ii) If x is the third proportional to .0238 and 7.805,

$$\begin{aligned} x \times .0238 &= (7.805)^2; \therefore x = \frac{(7.805)^2}{.0238} \\ 2 \log 7.805 &= 1.7848 \\ \log .0238 &= \bar{2}.3766 \\ \log x &= \bar{3}.4082; \text{ whence } x = 2560. \end{aligned}$$

28. Here $x = \{\sqrt[3]{347.3} \times \sqrt[5]{256.4}\}^{\frac{1}{10}}$
 $= (347.3)^{\frac{1}{6}} \times (256.4)^{\frac{1}{10}}.$

$$\begin{aligned} \frac{1}{6} \log 347.3 &= .4234 \\ \frac{1}{10} \log 256.4 &= .2409 \\ \log x &= .6643; \text{ whence } x = 4.616. \end{aligned}$$

29. $5 \log x + 3 \log y = \log 5,$
 $2 \log x + 7 \log y = \log 11.$

These equations give

$$\begin{aligned} \log x &= \frac{7 \log 5 - 3 \log 11}{29}, & \log y &= \frac{5 \log 11 - 2 \log 5}{29}. \\ 7 \log 5 &= 4.8930 & 5 \log 11 &= 5.2070 \\ 3 \log 11 &= 3.1242 & 2 \log 5 &= 1.3980 \\ 29) \overline{1.7688} & (.06099 & 29) \overline{3.8090} & (.1313 \\ & 288 & & 90 \\ & 27 & & 39 \\ & & & 100 \\ \therefore \log x &= .0610; & \therefore \log y &= .1313; \\ \text{whence } x &= 1.151. & \text{whence } y &= 1.353. \end{aligned}$$

30. $\log l = \log 2.863 = .4569$
 $\log g = \log 32.19 = 1.5077$

$$\begin{aligned} 2) \overline{2.9492} \\ \log \sqrt{\frac{l}{g}} &= \bar{1}.4746 \\ \log 2 &= .3010 \\ \log \pi &= .4972 \\ & \overline{.2728}; \\ \text{whence the required value} &= 1.874. \end{aligned}$$

31. $\log m = \log 18.34 = 1.2634$
 $\log v^2 = 2 \log 35.28 = 3.0950$
 $\qquad\qquad\qquad \underline{4.3584}$
 $\log 2 = \underline{.3010}$
 $\log \frac{1}{2} mv^2 = 4.0574; \text{ whence } \frac{1}{2} mv^2 = 11410.$

32. (i) $\log p = \log 93.75 = 1.9719$
 $\log r^n = 4 \log 1.03 = \underline{.0512}$
 $\log pr^n = 2.0231; \text{ whence } pr^n = 105.4.$

(ii) $\log r^3 = 3 \log 5.875 = 2.3070$ $\log 355 = 2.5502$
 $\log \pi = .4971$ $\log 113 = 2.0531$
 $\log 4 = \underline{.6021}$ $\log \pi = \underline{.4971}$
 $\qquad\qquad\qquad \underline{3.4062}$
 $\log 3 = \underline{.4771}$
 $\log \frac{4}{3} \pi r^3 = 2.9291; \text{ whence } \frac{4}{3} \pi r^3 = 849.4.$

33. $\log m = \log 33.47 = 1.5246$ $\log g = \log 32.19 = 1.5077$
 $\log v^2 = \log 3600 = 3.5563$ $\log r = \log 9.6 = .9823$
 $\qquad\qquad\qquad \underline{5.0809}$ $\qquad\qquad\qquad \underline{2.4900}$
 $\qquad\qquad\qquad 2.4900$
 $\log F = 2.5909; \text{ whence } F = 389.8.$

34. $r^3 = \frac{3 \times 537.6}{4 \times 3.1416}.$
 $\log 3 = .4771$ $\log 4 = .6021$
 $\log 537.6 = 2.7305$ $\log 3.1416 = \underline{.4971}$
 $\qquad\qquad\qquad \underline{3.2076}$ $\qquad\qquad\qquad \underline{1.0992}$
 $\qquad\qquad\qquad \underline{1.0992}$
 $\qquad\qquad\qquad 3) 2.1084$
 $\log r = .7028; \text{ whence } r = 5.044.$

35. $f = \frac{2s}{t^2} = \frac{578.6 \times 8^2}{31^2}.$
 $\log 578.6 = 2.7624$
 $\log 64 = 1.8062$
 $\qquad\qquad\qquad \underline{4.5686}$
 $2 \log 31 = 2.9828$
 $\log f = 1.5858; \text{ whence } f = 38.53.$

36. $n \log x + \log y = 8 + \log 8.7$
 $n \log 73.96 + \log 27.25 = 8 + \log 8.7 = 8.9395$
 $\log 27.25 = 1.4354$
 $\therefore n \log 73.96 = 7.5041.$
 $\therefore n = \frac{7.5041}{\log 73.96} = \frac{7.5041}{1.8690}$ $18.69) 75041 (4.015$
 $\qquad\qquad\qquad \underline{281}$
 $\qquad\qquad\qquad \underline{94}$
 $\qquad\qquad\qquad \underline{1}$
 $= 4.015.$

$$37. \quad r^3 = \frac{3V}{4\pi} = \frac{2 \times 33.87}{4 \times 3.1416}.$$

$$\begin{array}{r} \log 3 = .4771 \\ \log 33.87 = 1.5298 \\ \hline 2.0069 \\ \hline 1.0992 \\ 3) .9077 \end{array} \quad \begin{array}{r} \log 4 = .6021 \\ \log 3.1416 = .4971 \\ \hline 1.0992 \end{array}$$

$$\log r = .3026; \text{ whence } r = 2.007.$$

38. Let d be the diameter; then

$$\frac{4}{3} \pi \left(\frac{d}{2} \right)^3 = (36.4)^3;$$

$$\therefore d^3 = \frac{6 \times (36.4)^3}{\pi};$$

$$\therefore 3 \log d = \log 6 + 3 \log 36.4 - \log \pi.$$

$$\text{Thus } d = 45.16 \text{ cm.}$$

$$\begin{array}{r} \log 6 = .7782 \\ 3 \log 36.4 = 4.6833 \\ \hline 5.4615 \\ \log \pi = .4971 \\ 3) 4.9644 = 3 \log d \\ \hline 1.6548 \\ \text{antilog } 1.6548 = 45.16. \end{array}$$

$$\begin{array}{r} 39. \quad 2 \log v = \log r + \log g - \log 289 \\ \quad \quad \quad = \log 4000 + \log 32.2 - \log 5280 - \log 289. \\ \log 4000 = 3.6021 \quad \log 5280 = 3.7226 \\ \log 32.2 = 1.5079 \quad \log 289 = 2.4609 \\ \hline 5.1100 \\ \hline 6.1835 \\ 2) 2.9265 \end{array}$$

$$\log v = 1.4632; \text{ whence } v = 2905.$$

$$\text{Let } E = \frac{2\pi r}{v \times 60^2}; \text{ then } \log E = \log 2\pi r - \log v - 2 \log 60$$

$$\begin{array}{r} \log 2 = .3010 \\ \log \pi = .4971 \\ \log r = 3.6021 \\ \hline 4.4002 \\ \hline 3.0196 \end{array} \quad \begin{array}{r} \log v = 1.4632 \\ 2 \log 60 = 3.5564 \\ \hline 3.0196 \end{array}$$

$$\log E = 1.3806; \text{ whence } E = 24, \text{ approximately.}$$

EXAMPLES. XV. e. PAGE 163 H.

In Examples 16—19 let the expression be denoted by x .

- | | |
|--|--|
| <p>16. $\log \sin 27^\circ 13' = 1.6602$
 $\log \cos 46^\circ 16' = 1.8397$
 $\log x = 1.4999;$
 whence $x = .3161$.</p> | <p>17. $\log \sin 47^\circ 13' = 1.8656$
 $\log \tan 22^\circ 27' = 1.6162$
 $\log x = .2494;$
 whence $x = 1.776$.</p> |
| <p>18. $\log \sin 34^\circ 17' = 1.7507$
 $\log \tan 82^\circ 6' = .8577$
 $\hline .6084$
 $\log \cos 12^\circ 37' = 1.9894$
 $\log x = .6190;$
 whence $x = 4.159$.</p> | <p>19. $x = \cos 28^\circ 14' \times \cos 37^\circ 26'.$
 $\log \cos 28^\circ 14' = 1.9450$
 $\log \cos 37^\circ 26' = 1.8998$
 $\hline 1.8448;$
 whence $x = .6995$.</p> |

$$\begin{aligned}
 20. \quad 7 \log \tan x &= \log 11 - \log 13. \\
 \log 11 &= 1.0414 \\
 \log 13 &= 1.1139 \\
 7) \quad \overline{1.9275} \\
 \log \tan x &= \overline{1.9896}; \\
 \text{whence } x &= 44^\circ 19'.
 \end{aligned}$$

$$\begin{aligned}
 21. \quad \log 32.73 &= 1.5149 \\
 \log 27.86 &= 1.4449 \\
 \log \sin 30^\circ 16' &= \overline{1.7025} \\
 \log ab \sin C &= \overline{2.6623}; \\
 \text{whence } ab \sin C &= 459.5.
 \end{aligned}$$

$$\begin{aligned}
 22. \quad (i) \quad na^2 \cot \frac{\pi}{n} &= 32 \cot 22\frac{1}{2}^\circ \\
 &= 32 \tan 67\frac{1}{2}^\circ \\
 &= 32 \times 19.3136 \\
 &= 77.2544.
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad \frac{nr^2}{2} \sin \frac{2\pi}{n} &= 5 \times (3.3)^2 \sin 36^\circ. \\
 \log 5 &= .6990 \\
 2 \log 3.3 &= 1.0370 \\
 \log \sin 36^\circ &= \overline{1.7692} \\
 &\quad \overline{1.5052}; \\
 \therefore \text{required value} &= 32.00.
 \end{aligned}$$

$$\begin{aligned}
 23. \quad \tan \phi &= \frac{.7}{1 - (.35)^2} \sin 56^\circ 14' \\
 &= \frac{.7}{1.35 \times .65} \sin 56^\circ 14'.
 \end{aligned}$$

$$\begin{aligned}
 \log .7 &= \overline{1.8451} \\
 \log \sin 56^\circ 14' &= \overline{1.9198} \\
 \overline{1.7649} \\
 \overline{1.9432} \\
 \overline{1.8217}; \text{ whence } \phi &= 33^\circ 33'.
 \end{aligned}$$

$$\begin{aligned}
 \log 1.35 &= .1303 \\
 \log .65 &= \overline{1.8129} \\
 \overline{1.9432}
 \end{aligned}$$

$$24. \quad l = \frac{2 \times 32.78 \times 19.23}{52.01} \cos 57^\circ 47'.$$

$$\begin{aligned}
 \log 32.78 &= 1.5156 \\
 \log 38.46 &= 1.5850 \\
 \log \cos 57^\circ 47' &= \overline{1.7268} \\
 &\quad \overline{2.8274} \\
 \log 52.01 &= \overline{1.7161} \\
 \log l &= \overline{1.1113}; \text{ whence } l = 12.92.
 \end{aligned}$$

$$25. \quad \text{Expression} = \frac{2 \times (48)^2 \sin 23^\circ}{32.19 \times (.63)^2 \cos^2 23^\circ}.$$

$$\begin{aligned}
 \log 2 &= .3010 \\
 2 \log 48 &= 3.3624 \\
 \log \sin 23^\circ &= \overline{1.5919} \\
 \overline{3.2553} \\
 \overline{1.0343} \\
 \overline{2.2210}; \text{ whence value of expression} &= 166.3.
 \end{aligned}$$

$$\begin{aligned}
 \log 32.19 &= 1.5077 \\
 2 \log .63 &= \overline{1.5986} \\
 2 \log \cos 23^\circ &= \overline{1.9280} \\
 \overline{1.0343}
 \end{aligned}$$

EXAMPLES. XVI. a. PAGE 166.

$$1. \text{ First side} = \frac{s(s-a)}{c} + \frac{s(s-b)}{c} = \frac{s(2s-a-b)}{c} = s.$$

$$2. \text{ First side} = s \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \cdot \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \\ = s \sqrt{\frac{(s-a)^2}{s^2}} = s-a.$$

$$3. \text{ First side} = \frac{1-\cos A}{1+\cos B} = \frac{2\sin^2 \frac{A}{2}}{2\sin^2 \frac{B}{2}} = \frac{(s-b)(s-c)}{bc} \times \frac{ca}{(s-c)(s-a)} \\ = \frac{a(s-b)}{c(s-a)} = \frac{a(a+c-b)}{b(b+c-a)}.$$

$$4. \text{ First side} = \frac{b(s-b)(s-c)}{bc} + \frac{a(s-c)(s-a)}{ca} \\ = \frac{(s-c)\{s-b+s-a\}}{c} = \frac{c(s-c)}{c} = s-c.$$

$$5. \text{ Each of the expressions reduces to } \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$6. \tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \sqrt{\frac{3 \times 14}{24 \times 7}} = \frac{1}{2}.$$

$$7. \cot \frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \sqrt{\frac{21 \times 6}{8 \times 7}} = \frac{3}{2}.$$

$$8. \text{ First side} = \frac{s(s-a) + s(s-b) + s(s-c)}{abc} \\ = \frac{s\{3s-(a+b+c)\}}{abc} = \frac{s^2}{abc}.$$

$$9. \text{ First side} = \frac{b-c}{a} \cdot \frac{s(s-a)}{bc} + \text{two similar terms} \\ = \frac{(b-c)(s^2-as)}{abc} + \text{two similar terms} \\ = \frac{s^2\{(b-c)+(c-a)+(a-b)\} - s\{a(b-c)+b(c-a)+c(a-b)\}}{abc} \\ = 0.$$

EXAMPLES. XVI. b. PAGE 169.

$$1. \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}} = \sqrt{\frac{7 \times 4}{5 \times 8}} = \sqrt{\frac{7}{10}}.$$

$$\log \sin \frac{C}{2} = \frac{1}{2} (\log 7 - 1)$$

$$= \bar{1} \cdot 9225490$$

$$\log \sin 56^\circ 47' = \bar{1} \cdot 9225205$$

$$\text{diff.} \quad \quad \quad 285$$

$$\text{prop}^l. \text{ increase} = \frac{285}{827} \times 60'' = 20 \cdot 6'';$$

$$\therefore \frac{C}{2} = 56^\circ 47' 20 \cdot 6'', \text{ and } C = 113^\circ 34' 41''.$$

$$\begin{array}{r} 285 \\ 60 \\ 827 \overline{) 17100} \text{ (} 20 \cdot 6 \\ 1654 \\ \hline 5600 \\ 4962 \end{array}$$

$$2. \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{16 \times 24}{67 \times 27}} = \sqrt{\frac{128}{603}}.$$

$$\log \tan \frac{A}{2} = \bar{1} \cdot 6634464$$

$$= \log \tan 24^\circ 44' 13'';$$

$$\therefore A = 49^\circ 28' 26''.$$

$$\begin{array}{r} \log 128 = 2 \cdot 1072100 \\ \log 603 = 2 \cdot 7803173 \\ \hline 2) \bar{1} \cdot 3268927 \\ \hline \bar{1} \cdot 6634464 \end{array}$$

$$3. \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}} = \sqrt{\frac{15 \times 5}{2 \times 2 \times 4 \times 6}} = \frac{5}{4\sqrt{2}}.$$

$$\log \cos \frac{B}{2} = \log 5 - \frac{5}{2} \log 2$$

$$= \bar{1} \cdot 9463950$$

$$\log \cos 27^\circ 53' = \bar{1} \cdot 9464040$$

$$\text{diff.} \quad \quad \quad 90$$

$$\text{prop}^l. \text{ increase} = \frac{90}{669} \times 60'' = 8 \cdot 07'';$$

$$\therefore \frac{B}{2} = 27^\circ 53' 8 \cdot 07'', \text{ and } B = 55^\circ 46' 16''.$$

$$\begin{array}{r} \log 5 = \cdot 6989700 \\ \frac{5}{2} \log 2 = \cdot 7525750 \\ \hline \bar{1} \cdot 9463950 \\ \hline 30 \\ 60 \\ 223 \overline{) 1800} \text{ (} 8 \cdot 07 \\ 1784 \\ \hline 1600 \end{array}$$

$$4. \quad \cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}} = \sqrt{\frac{9 \times 2}{5 \times 6}} = \sqrt{\frac{6}{10}}.$$

$$\log \cos \frac{C}{2} = \frac{1}{2} (\log 6 - 1)$$

$$= \bar{1}.8890757$$

$$\log \cos 39^\circ 14' = \bar{1}.8890644$$

$$\text{diff.} \quad \frac{113}{113}$$

$$\text{prop}^l. \text{ decrease} = \frac{113}{1032} \times 60'' = 6.6'';$$

$$\therefore \frac{C}{2} = 39^\circ 13' 53.4'', \text{ and } C = 78^\circ 27' 47''.$$

$$\begin{array}{r} 113 \\ 60 \\ 1032 \overline{) 67800} \quad (6.6 \\ \underline{6192} \\ 5880 \\ \underline{6192} \end{array}$$

$$5. \quad \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{.75 \times 2}{3.75}} = \sqrt{\frac{2^2}{10}}.$$

$$\log \tan \frac{C}{2} = \frac{1}{2} \{2 \log 2 - 1\} = \frac{1}{2} (\bar{1}.6020600)$$

$$= \bar{1}.8010300$$

$$\log \tan 32^\circ 18' = \bar{1}.8008365$$

$$\text{diff.} \quad \frac{1935}{1935}$$

$$\text{prop}^l. \text{ increase} = \frac{1935}{2796} \times 60'' = 41.5'';$$

$$\therefore \frac{C}{2} = 32^\circ 18' 41.5'', \text{ and } C = 64^\circ 37' 23''.$$

$$\begin{array}{r} 1935 \\ 60 \\ 2796 \overline{) 116100} \quad (41.5 \\ \underline{11184} \\ 4260 \\ \underline{2796} \\ 14640 \end{array}$$

$$6. \quad \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{3 \times 4}{15 \times 8}} = \sqrt{\frac{1}{10}}.$$

$$\log \tan \frac{C}{2} = -\frac{1}{2} = \bar{1}.500000$$

$$\log \tan 17^\circ 33' = \bar{1}.500042$$

$$\text{diff.} \quad \frac{42}{42}$$

$$\text{prop}^l. \text{ decrease} = \frac{42}{439} \times 60'' = 5.7'';$$

$$\therefore \frac{C}{2} = 17^\circ 32' 54.3'', \text{ and } C = 35^\circ 5' 49''.$$

$$\begin{array}{r} 42 \\ 60 \\ 439 \overline{) 2520} \quad (5.7 \\ \underline{2195} \\ 3250 \end{array}$$

7. Let $a=4$, $b=10$, $c=11$.

$$\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}} = \sqrt{\frac{25}{2} \cdot \frac{3}{2} \cdot \frac{1}{40}} = \sqrt{\frac{15}{2^5}} = \sqrt{\frac{30}{2^6}}.$$

$$\log \cos \frac{C}{2} = \frac{1}{2} (\log 3 + 1 - 6 \log 2)$$

$$= \bar{1}.8354707$$

$$\log \cos 46^\circ 47' = \bar{1}.8355378$$

$$\text{diff.} \quad \underline{671}$$

$$\text{prop}^l. \text{ increase} = \frac{671}{1345} \times 60'' = 30''.$$

$$\therefore \frac{C}{2} = 46^\circ 47' 30'', \text{ and } C = 93^\circ 35'.$$

$$\begin{array}{r} 1 + \log 3 = 1.4771213 \\ 6 \log 2 = 1.8061800 \\ \hline 2) \bar{1}.6709413 \\ \hline \bar{1}.8354707 \end{array}$$

$$8. \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{63 \times 7}{21 \times 8}} = \frac{1}{2}.$$

$$\log \tan \frac{B}{2} = -\log 2 = \bar{1}.6989700$$

$$\log \tan 26^\circ 33' = \bar{1}.6986847$$

$$\text{diff.} \quad \underline{2853}$$

$$\text{prop}^l. \text{ increase} = \frac{2853}{3159} \text{ of } 60'' = 54.2''.$$

$$\therefore \frac{B}{2} = 26^\circ 33' 54.2'', \text{ and } B = 53^\circ 7' 48''.$$

$$\begin{array}{r} 2853 \\ 60 \\ 3159) 171180 (54.2 \\ \hline 15795 \\ 13230 \\ 12636 \\ \hline 5940 \end{array}$$

Again $\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{6 \times 8}{21 \times 7}} = \frac{4}{7}.$

$$\log \tan \frac{C}{2} = 2 \log 2 - \log 7$$

$$= \bar{1}.7569620$$

$$\log \tan 29^\circ 44' = \bar{1}.7567587$$

$$\text{diff.} \quad \underline{2033}$$

$$\text{prop}^l. \text{ increase} = \frac{2033}{2933} \times 60'' = 41.5''.$$

$$\therefore \frac{C}{2} = 29^\circ 44' 41.5'', \text{ and } C = 59^\circ 29' 23''.$$

$$\therefore A = 67^\circ 22' 49''.$$

$$\begin{array}{r} 2 \log 2 = .6020600 \\ \log 7 = .8450980 \\ \hline \bar{1}.7569620 \end{array}$$

$$\begin{array}{r} 2033 \\ 60 \\ 2933) 121980 (41.5 \\ \hline 11732 \\ 4660 \\ 2933 \\ \hline 17270 \end{array}$$

$$9. \quad \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{\frac{3}{2} \times \frac{5}{2} \times \frac{2}{9} \times 2}{s(s-b)}} = \sqrt{\frac{5}{3}}.$$

$$\log \tan \frac{B}{2} = \frac{1}{2} (\log 5 - \log 3)$$

$$= \cdot 1109244$$

$$\log \tan 52^\circ 14' = \cdot 1108395$$

$$\text{diff.} \quad \quad \quad 849$$

$$\log 5 = \cdot 6989700$$

$$\log 3 = \cdot 4771213$$

$$2 \quad \cdot 2218487$$

$$\cdot 1109244$$

$$\text{prop}^l. \text{ increase} = \frac{849}{435} \times 10'' = 19 \cdot 5'';$$

$$\therefore \frac{B}{2} = 52^\circ 14' 19 \cdot 5'', \text{ and } B = 104^\circ 28' 39''.$$

$$\text{Again} \quad \tan \frac{C}{2} = \sqrt{\frac{(s-b)(s-a)}{s(s-c)}} = \sqrt{\frac{\frac{1}{2} \times \frac{3}{2} \times \frac{2}{9} \times \frac{2}{5}}{s(s-c)}} = \sqrt{\frac{1}{3 \times 5}}.$$

$$\log \tan \frac{C}{2} = -\frac{1}{2} (\log 3 + \log 5)$$

$$= \bar{1} \cdot 4119544$$

$$\log \tan 14^\circ 28' = \bar{1} \cdot 4116146$$

$$\text{diff.} \quad \quad \quad 3398$$

$$\text{prop}^l. \text{ increase} = \frac{3398}{870} \times 10'' = 39'';$$

$$\therefore \frac{C}{2} = 14^\circ 28' 39'', \text{ and } C = 28^\circ 57' 18''.$$

$$\therefore A = 46^\circ 34' 3''.$$

EXAMPLES. XVI. c. PAGE 173.

$$1. \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{1}{5} \cot 30^\circ = \frac{2}{10} \sqrt{3}.$$

$$\log \tan \frac{A-B}{2} = \log 2 + \frac{1}{2} \log 3 - 1$$

$$= \bar{1} \cdot 5395907$$

$$\log \tan 19^\circ 6' = \bar{1} \cdot 5394287$$

$$\text{diff.} \quad \quad \quad 1620$$

$$\log 2 = \cdot 3010300$$

$$\frac{1}{2} \log 3 = \cdot 2385607$$

$$\cdot 5395907$$

$$\text{prop}^l. \text{ increase} = \frac{1620}{4084} \times 60'' = 24''.$$

$$\therefore \frac{A-B}{2} = 19^{\circ} 6' 24'', \text{ and } \frac{A+B}{2} = 60^{\circ};$$

$$\therefore A = 79^{\circ} 6' 24''; B = 40^{\circ} 53' 36''.$$

2.

$$\tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2} = \frac{8}{10} \cot 32^{\circ} 30'.$$

$$\log \tan \frac{C-A}{2} = 3 \log 2 - 1 + \log \cot 32^{\circ} 30'$$

$$= .0989027$$

$$\log \tan 51^{\circ} 28' = .0988763$$

$$\text{diff.} \quad \quad \quad 264$$

$$\text{prop}^l. \text{ increase} = \frac{264}{2592} \times 60'' = 6'';$$

$$\therefore \frac{C-A}{2} = 51^{\circ} 28' 6'',$$

$$\frac{C+A}{2} = 57^{\circ} 33';$$

$$\therefore C = 108^{\circ} 58' 6''; A = 6^{\circ} 1' 54''.$$

$$\begin{array}{r} 3 \log 2 = .9030900 \\ \log \cot 32^{\circ} 30' = .1958127 \\ \hline 1.0989027 \end{array}$$

3.

$$\tan \frac{B-A}{2} = \frac{b-a}{b+a} \cot \frac{C}{2} = \frac{5}{12} \sqrt{3}.$$

$$\log \tan \frac{B-A}{2} = 1 - 3 \log 2 - \frac{1}{2} \log 3$$

$$= \bar{1}.8583494$$

$$\log \tan 35^{\circ} 49' = \bar{1}.8583357$$

$$\text{diff.} \quad \quad \quad 137$$

$$\text{prop}^l. \text{ increase} = \frac{137}{2662} \times 60'' = 3'';$$

$$\therefore \frac{B-A}{2} = 35^{\circ} 49' 3'',$$

$$\frac{B+A}{2} = 60^{\circ};$$

$$\therefore B = 95^{\circ} 49' 3''; A = 24^{\circ} 10' 57''.$$

$$\begin{array}{r} \frac{1}{2} \log 3 = .2385606 \\ 3 \log 2 = .9030900 \\ \hline 1.1416506 \end{array}$$

$$4. \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{4}{50} \cot 22^\circ 15' = \frac{8}{100} \cot 22^\circ 15'.$$

$$\log \tan \frac{B-C}{2} = \bar{1}.2912491$$

$$\log \tan 11^\circ 3' = \bar{1}.2906713$$

$$\text{diff.} \quad \underline{5778}$$

$$\text{prop}^l. \text{ increase} = \frac{5778}{6711} \times 60'' = 52'';$$

$$3 \log 2 - 2 = \bar{2}.9030900$$

$$\log \cot 22^\circ 15' = \cdot 3881591$$

$$\underline{\bar{1}.2912491}$$

$$\therefore \frac{B-C}{2} = 11^\circ 3' 52'', \text{ and } \frac{B+C}{2} = 67^\circ 45';$$

$$\therefore B = 78^\circ 48' 52''; C = 56^\circ 41' 8''.$$

$$5. \quad \tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2} = \frac{10}{32} \cot 17^\circ 21' 15''.$$

$$\log \tan \frac{C-A}{2} = 1 + \log \cot 17^\circ 21' 15'' - 5 \log 2$$

$$= 1.5051500 - 1.5051500$$

$$= 0.$$

$$\therefore \frac{C-A}{2} = 45^\circ, \text{ and } \frac{C+A}{2} = 72^\circ 38' 45'';$$

$$\therefore C = 117^\circ 38' 45''; A = 27^\circ 38' 45''.$$

$$6. \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{1}{4} \cot 30^\circ 15'.$$

$$\log \tan \frac{A-B}{2} = -2 \log 2 + \log \cot 30^\circ 15'$$

$$= \bar{1}.63214$$

$$\log \tan 23^\circ 13' = \bar{1}.63240$$

$$\text{diff.} \quad \underline{26}$$

$$\log \cot 30^\circ 15' = \cdot 23420$$

$$2 \log 2 = \cdot 60206$$

$$\underline{1.63214}$$

$$\text{prop}^l. \text{ decrease} = \frac{26}{35} \times 60'' = 45'';$$

$$\therefore \frac{A-B}{2} = 23^\circ 12' 15'', \text{ and } \frac{A+B}{2} = 59^\circ 45';$$

$$\therefore A = 82^\circ 57' 15''; B = 36^\circ 32' 45''.$$

7. $\tan \frac{A-C}{2} = \frac{a-c}{a+c} \cot \frac{B}{2} = \frac{71}{283} \cot 28^\circ 14'.$

$\log \tan \frac{A-C}{2} = \overline{1.3556602}$	$\log 71 = 1.8512583$
$\log \tan 12^{\circ} 46' = \overline{1.3552267}$	$\log \cot \frac{B}{2} = \frac{.2700705}{2.1218288}$
diff. $\frac{4335}{4335}$	$\log 583 = \frac{2.7656686}{1.3556602}$
prop ^l . increase = $\frac{4335}{5859} \times 60'' = 44''$;	

$$\therefore \frac{A-C}{2} = 12^\circ 46' 44'', \text{ and } \frac{A+C}{2} = 61^\circ 46';$$

$$\therefore A = 74^\circ 32' 44''; C = 48^\circ 59' 16''.$$

8. $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{3}{5} \cot 32^\circ 30'.$

$\log \tan \frac{B-C}{2} = \overline{1.9739640}$	$\log 3 = \cdot 4771213$
$\log \tan 43^\circ 18' = \overline{1.9742133}$	$\log 5 = \cdot 6989700$
diff. $\frac{2493}{2493}$	$\overline{1.7781513}$
$\text{prop}^1. \text{ decrease} = \frac{2493}{2531} \times 60'' = 59'';$	$\log \cot 32^\circ 30' = \cdot 1958127$
	$\overline{1.9739640}$

$$\therefore \frac{B-C}{2} = 43^\circ 17' 1'', \text{ and } \frac{B+C}{2} = 57^\circ 30';$$

$$\therefore B = 100^\circ 47' 1''; C = 14^\circ 12' 59''.$$

9. Here $\cot \frac{A-B}{2} = \frac{a+b}{a-b} \tan \frac{C}{2} = 2 \tan \frac{C}{2}$.

$$\begin{array}{l} \therefore \log \cot \frac{A-B}{2} = \log 2 + \log \tan 15^{\circ} 5' 2.5'' \\ \qquad \qquad \qquad = \bar{1}.7316236 \qquad \qquad \qquad \log 2 = .3010300 \\ \log \cot 61^{\circ} 41' = \bar{1}.7314436 \qquad \log \tan 15^{\circ} 5' = \bar{1}.4305727 \\ \text{diff.} \qquad \qquad \qquad \frac{\quad}{1800} \qquad \qquad \qquad \frac{2.5}{10} \times 838 = \frac{209}{\quad} \\ \text{prop}^{\text{l}}. \text{ decrease} = \frac{1800}{504} \times 10'' = 35.7''; \qquad \qquad \qquad \frac{\quad}{\bar{1}.7316236} \end{array}$$

$$\therefore \frac{A-B}{2} = 61^\circ 40' 24.3'', \text{ and } \frac{A+B}{2} = 74^\circ 54' 51.5'';$$

$$\therefore A = 136^\circ 35' 21.8''; B = 13^\circ 14' 33.2''.$$

EXAMPLES. XVI. d. PAGE 174.

1. Here
- $A = 180^\circ - 114^\circ 45' = 65^\circ 15'$
- .

$$c = \frac{a \sin C}{\sin A} = \frac{100 \sin 54^\circ 30'}{\sin 65^\circ 15'}.$$

$$\begin{aligned} \log c &= 1.9525317 \\ &= \log 89.646162; \\ \therefore c &= 89.646162. \end{aligned}$$

$$\begin{aligned} \log \sin 54^\circ 30' &= \bar{1}.9106860 \\ \log 100 &= 2 \\ \hline &1.9106860 \\ \log \sin 65^\circ 15' &= \bar{1}.9581543 \\ \hline &1.9525317 \end{aligned}$$

2. $a = \frac{c \sin A}{\sin C} = \frac{270 \sin 55^\circ}{\sin 60^\circ} = 270 \sin 55^\circ \times \frac{2}{\sqrt{3}}.$

$$\therefore \log a = 1 + 3 \log 3 + \log \sin 55^\circ - \frac{1}{2} \log 3 + \log 2$$

$$\begin{aligned} &= 2.4071977 \\ \log 255.38 &= 2.4071869 \\ \text{diff.} &= 108 \\ \therefore \text{prop}^l. \text{ increase} &= \frac{108}{170} \times .01 = .0064; \\ \therefore a &= 255.3864. \end{aligned}$$

$$\begin{aligned} \log 270 &= 2.4313639 \\ \log \sin 55 &= \bar{1}.9133645 \\ \log 2 &= .3010300 \\ \hline &2.6457584 \\ \frac{1}{2} \log 3 &= .2385607 \\ \hline &2.4071977 \end{aligned}$$

3. $c = \frac{b \sin C}{\sin B} = \frac{100 \sin 62^\circ 5'}{\sin 72^\circ 14'}.$

$$\begin{aligned} \log c &= 1.96749 \\ &= \log 92.788; \\ \therefore c &= 92.788. \end{aligned}$$

$$\begin{aligned} \log \sin 62^\circ 5' &= \bar{1}.94627 \\ \log 100 &= 2 \\ \hline &1.94627 \\ \log \sin 72^\circ 14' &= \bar{1}.97878 \\ \hline &1.96749 \end{aligned}$$

4. Here
- $A = 180^\circ - 148^\circ 40' = 31^\circ 20'$
- .

$$b = \frac{a \sin B}{\sin A} = \frac{102 \sin 70^\circ 30'}{\sin 31^\circ 20'}.$$

$$\begin{aligned} \log b &= 2.267 \\ &= \log 185; \\ \therefore b &= 185. \end{aligned}$$

$$\begin{aligned} \log 102 &= 2.009 \\ \log \sin 70^\circ 30' &= \bar{1}.974 \\ \hline &1.983 \\ \log \sin 31^\circ 20' &= \bar{1}.716 \\ \hline &2.267 \end{aligned}$$

$$\text{Again } c = \frac{a \sin C}{\sin A},$$

$$\log c = 2.283$$

$$= \log 192;$$

$$\therefore c = 192.$$

$$\log 102 = 2.009$$

$$\log \sin 78^\circ 10' = \bar{1}.990$$

$$\bar{1}.999$$

$$\log \sin 31^\circ 20' = \bar{1}.716$$

$$2.283$$

$$5. \text{ Here } c = \frac{a \sin C}{\sin A} = \frac{123}{\sqrt{2} \sin 15^\circ 43'}.$$

$$\log c = 2.5066124$$

$$\log 321.10 = 2.5066403$$

$$\text{diff.} \quad \frac{279}{279}$$

$$\text{prop}^l. \text{ decrease} = \frac{279}{135} \times .01 = .02066.$$

$$\text{Thus } c = 321.0793.$$

$$\frac{1}{2} \log 2 = .1505150$$

$$\log \sin 15^\circ 43' = \bar{1}.4327777$$

$$\bar{1}.5832927$$

$$\log 123 = 2.0899051$$

$$2.5066124$$

$$6. a = \frac{b \sin A}{\sin B} = \frac{1006.62 \sin 44^\circ}{\sin 66^\circ}.$$

$$\log 1006.62 = 3.0028656$$

$$\log \sin 44^\circ = \bar{1}.8417713$$

$$2.8446369$$

$$\log \sin 66^\circ = \bar{1}.9607302$$

$$\log a = 2.8839067$$

$$\therefore a = 765.4321.$$

$$c = \frac{b \sin C}{\sin B} = \frac{1006.62 \sin 70^\circ}{\sin 66^\circ}.$$

$$\log 1006.62 = 3.0028656$$

$$\log \sin 70^\circ = \bar{1}.9729858$$

$$2.9758514$$

$$\log \sin 66^\circ = \bar{1}.9607302$$

$$\log c = 3.0151212$$

$$\therefore c = 1035.43.$$

$$7. \text{ Here } A = \text{supplement of } 75^\circ 45';$$

$$\therefore b = \frac{1652 \sin 26^\circ 30'}{\sin 75^\circ 45'}.$$

$$\log b = 2.8852436$$

$$\log 767.80 = 2.8852481$$

$$\text{diff.} \quad \frac{45}{45}$$

$$\text{prop}^l. \text{ decrease} = \frac{45}{57} \times .01 = .008; \quad \therefore b = 767.792.$$

$$\log 1652 = 3.2180100$$

$$\log \sin 26^\circ 30' = \bar{1}.6495274$$

$$2.8675374$$

$$\log \sin 73^\circ 45' = \bar{1}.9822938$$

$$2.8852436$$

$$\text{Again } c = \frac{1652 \sin 47^\circ 15'}{\sin 73^\circ 45'};$$

$$\log c = 3.1016030$$

$$\log 1263.6 = 3.1016096$$

$$\text{diff.} \quad \frac{66}{66}$$

$$\text{prop}^l. \text{ decrease} = \frac{66}{344} \times .1 = .019; \quad \therefore c = 1263.58.$$

$$\log 1652 = 3.2180100$$

$$\log \sin 47^\circ 15' = \bar{1}.8658868$$

$$3.0838968$$

$$\log \sin 73^\circ 45' = \bar{1}.9822938$$

$$3.1016030$$

EXAMPLES. XVI. e. PAGE 176.

$$1. \quad \sin A = \frac{a \sin B}{b} = \frac{145}{178} \sin 41^\circ 10'.$$

$\log \sin A = \bar{1}.7293399,$ $\therefore A = 32^\circ 25' 35''.$		$\log 145 = 2.1613680$ $\log \sin 41^\circ 10' = \bar{1}.8183919$ $\underline{1.9797599}$ $\log 178 = 2.2504200$ $\underline{\bar{1}.7293399}$
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$$2. \quad \sin B = \frac{b}{a} \sin A = \frac{127}{85} \sin 26^\circ 26'.$$

$\log \sin B = \bar{1}.8228972,$ $\therefore B = 41^\circ 41' 28'';$ and since a is $< b$, there is another value of B , namely, $B = 138^\circ 18' 32''.$		$\log 127 = 2.1038037$ $\log \sin 26^\circ 26' = \bar{1}.6485124$ $\underline{1.7523161}$ $\log 85 = 1.9294189$ $\underline{\bar{1}.8228972}$
---	--	---

$$3. \quad \sin B = \frac{b}{c} \sin C = \frac{4}{5} \sin 45^\circ = \frac{8}{10} \cdot \frac{1}{\sqrt{2}},$$

$$\log \sin B = 3 \log 2 - 1 - \frac{1}{2} \log 2.$$

$\therefore \log \sin B = \bar{1}.7525750.$ $\therefore B = 34^\circ 26'$ and $A = 100^\circ 34'.$		$3 \log 2 = .9030900$ $1 + \frac{1}{2} \log 2 = 1.1505150$ $\underline{\bar{1}.7525750}$
--	--	--

$$4. \quad \sin B = \frac{b}{a} \sin A = \frac{1706}{1405} \sin 40^\circ.$$

$\log \sin B = \bar{1}.8923702$ $\log \sin 51^\circ 18' = \bar{1}.8923342$ diff. 360 $\text{prop}^l. \text{ increase} = \frac{360}{1012} \times 60'' = 21''.$		$\log 1706 = 3.2319790$ $\log \sin 40^\circ = \bar{1}.8080675$ $\underline{3.0400465}$ $\log 1405 = 3.1476763$ $\underline{\bar{1}.8923702}$
---	--	--

$\therefore B = 51^\circ 18' 21''$; but since $a < b$, there is another value of B , namely, $128^\circ 41' 39''.$

$$5. \quad \sin C = \frac{c}{b} \sin B = \frac{394}{573} \sin 112^\circ 4' = \frac{394}{573} \cos 22^\circ 4'.$$

$$\therefore \log \sin C = \bar{1}.8043030$$

$$\log \sin 39^\circ 35' = \bar{1}.8042757$$

$$\text{diff.} \quad \underline{\quad 273 \quad}$$

$$\text{prop}^l. \text{ increase} = \frac{273}{1527} \times 60'' = 11'';$$

$$\therefore C = 39^\circ 35' 11''; \text{ and } A = 28^\circ 20' 49''.$$

$$\log 394 = 2.5954962$$

$$\log \sin 112^\circ 4' = \bar{1}.9669614$$

$$\underline{\quad 2.5624576 \quad}$$

$$\log 573 = 2.7581546$$

$$\underline{\quad \bar{1}.8043030 \quad}$$

$$6. \quad \sin C = \frac{c}{b} \sin B = \frac{12}{8.4} \sin 37^\circ 36' = \frac{1}{7} \sin 37^\circ 36'.$$

$$\log \sin C = \bar{1}.9403352$$

$$\log \sin 60^\circ 39' = \bar{1}.9403381$$

$$\text{diff.} \quad \underline{\quad 29 \quad}$$

$$\text{prop}^l. \text{ decrease} = \frac{29}{711} \times 60'' = 2.4'.$$

$$\log \sin 37^\circ 36' = \bar{1}.7854332$$

$$\log .7 = \bar{1}.8450980$$

$$\underline{\quad \bar{1}.9403352 \quad}$$

$\therefore C = 60^\circ 38' 58''$; but since $b < c$, there is another value of C , namely, $119^\circ 21' 2''$. Thus $A = 81^\circ 45' 2''$, or $23^\circ 2' 58''$.

$$7. \quad (i) \quad \text{Here } \sin C = \frac{c \sin A}{a} = \frac{250}{125} \times \frac{1}{2} = 1.$$

$$\therefore C = 90^\circ, \text{ and there is no ambiguity.}$$

(ii) Here $\sin C = \frac{250}{200} \times \frac{1}{2} = \frac{5}{8}$, and since $a < c$ there will be two values of C satisfying the data.

(iii) Here $\sin C = \frac{125}{200} \times \frac{1}{2} = \frac{5}{16}$; but since $a > c$ there is only one solution.

From (ii) we have $\log \sin C = \log 5 - \log 8 = \bar{1}.79588$.

$$\therefore C = 38^\circ 41', \text{ or } 141^\circ 19'; \text{ and } A = 111^\circ 19', \text{ or } 8^\circ 41'.$$

Now in the obtuse-angled triangle we have

$$b = \frac{a \sin B}{\sin A} = \frac{200 \sin 8^\circ 41'}{\sin 30^\circ}.$$

$$\log b = 1.7809601$$

$$\log 60.389 = 1.7809578$$

$$\text{diff.} \quad \underline{\quad 23 \quad}$$

$$\text{prop}^l. \text{ increase} = \frac{23}{72} \times .001 = .0003.$$

$$\therefore b = 60.3893.$$

$$\log 200 = 2.3010300$$

$$\log \sin 8^\circ 41' = \bar{1}.1789001$$

$$\underline{\quad 1.4799301 \quad}$$

$$\log \sin 30^\circ = \bar{1}.6989700$$

$$\underline{\quad 1.7809601 \quad}$$

EXAMPLES. XVI. f. PAGE 180.

$$1. \quad \tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \sqrt{\frac{3 \cdot 4}{12 \cdot 5}} = \sqrt{\frac{1}{5}};$$

$$\therefore \log \tan \frac{B}{2} = -\frac{1}{2} \log 5 = -\frac{1}{2}(1 - \log 2)$$

$$= \bar{1} \cdot 6505150$$

$$\log \tan 24^\circ 5' = \bar{1} \cdot 6502809$$

$$\text{diff.} \quad \quad \quad 2341$$

$$\text{prop}^l. \text{ increase} = \frac{2341}{3390} \times 60'' = 41 \cdot 4'';$$

$$\therefore \frac{B}{2} = 24^\circ 5' 41 \cdot 4'', \text{ and } B = 48^\circ 11' 23''.$$

$$\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{4 \cdot 5}{12 \cdot 3}} = \sqrt{\frac{5}{3^2}};$$

$$\therefore \log \tan \frac{C}{2} = \frac{1}{2}(1 - \log 2 - 2 \log 3)$$

$$= \bar{1} \cdot 8723637$$

$$\log \tan 36^\circ 41' = \bar{1} \cdot 8721123$$

$$\text{diff.} \quad \quad \quad 2514$$

$$\text{prop}^l. \text{ increase} = \frac{2514}{2637} \times 60'' = 57 \cdot 2'';$$

$$\therefore \frac{C}{2} = 36^\circ 41' 57 \cdot 2'', \text{ and } C = 73^\circ 23' 54'';$$

$$\therefore A = 58^\circ 24' 43''.$$

2.

$$\cot \frac{A}{2} = \frac{b+c}{b-c} \tan \frac{B-C}{2}$$

$$= \frac{512}{162} \tan 12^\circ = \frac{256}{81} \tan 12^\circ.$$

$$\therefore \log \cot \frac{A}{2} = 8 \log 2 - 4 \log 3 + \log \tan 12^\circ$$

$$= \bar{1} \cdot 8272293 = \log \cot 56^\circ 6' 27'';$$

$$\therefore \frac{A}{2} = 56^\circ 6' 27'', \text{ and } A = 112^\circ 12' 54''.$$

$$\therefore B + C = 67^\circ 47' 6'', \text{ and } B - C = 24^\circ;$$

$$\therefore B = 45^\circ 53' 33'', \text{ and } C = 21^\circ 53' 33''.$$

$$3. \quad \sin A = \frac{2}{7}, \text{ if } A \text{ is the less of the two acute angles.}$$

$$\log \sin A = \log 2 - \log 7$$

$$= \bar{1}.455932$$

$$\log \sin 14^\circ 11' = \bar{1}.455921$$

$$\text{diff.} \quad \quad \quad \underline{11}$$

$$\text{prop}^l. \text{ increase} = \frac{11}{110} \times 60'' = 6''.$$

$$\therefore A = 14^\circ 11' 6'';$$

$$\therefore B = 90^\circ - 14^\circ 11' 6'' = 75^\circ 48' 54''.$$

$$4. \quad \text{Here } a = 2183, A = 30^\circ 22', B = 78^\circ 14', C = 71^\circ 24'.$$

$$b = \frac{a \sin B}{\sin A} = \frac{2183 \sin 78^\circ 14'}{\sin 30^\circ 22'}.$$

$$\log b = 3.6260817$$

$$\log 4227.4 = 3.6260733$$

$$\text{diff.} \quad \quad \quad \underline{84}$$

$$\text{prop}^l. \text{ increase} = \frac{84}{103} \times 1 = .0815;$$

$$\therefore b = 4227.4815.$$

$$\log 2183 = 3.3390537$$

$$\log \sin B = \bar{1}.9907766$$

$$\underline{3.3298303}$$

$$\log \sin A = \bar{1}.7037486$$

$$\underline{3.6260817}$$

$$5. \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{1}{9} \cot 11^\circ 10'.$$

$$\therefore \log \tan \frac{B-C}{2} = \log \cot 11^\circ 10' - 2 \log 3$$

$$= .70465 - .95424 = \bar{1}.75041.$$

$$\therefore \frac{B-C}{2} = 29^\circ 22' 26'', \text{ and } \frac{B+C}{2} = 78^\circ 50';$$

$$\therefore B = 108^\circ 12' 26''; C = 49^\circ 27' 34''.$$

Now

$$a = \frac{c \sin A}{\sin C};$$

$$\therefore \log a = \log 2 + \log \sin 22^\circ 29' - \log \sin 49^\circ 27' 34''$$

$$= .30103 + \bar{1}.57977 - \bar{1}.88079$$

$$= .00001;$$

$$\therefore a = 1, \text{ approximately.}$$

$$6. \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = .56234 \cot 29^{\circ} 21' 3''.$$

Now

$$\log \cot 29^{\circ} 21' = .250015$$

$$\log \cot 29^{\circ} 22'' = .249715$$

$$\text{diff. for } 60'' \quad \underline{300}$$

$$\therefore \text{prop}^l. \text{ decrease for } 3'' = \frac{3}{60} \times 300 = 15;$$

$$\therefore \log \cot 29^{\circ} 21' 3'' = .250000.$$

$$\therefore \log \tan \frac{A-B}{2} = \bar{1}.75 + .25 = 0.$$

$$\therefore \tan \frac{A-B}{2} = 1, \text{ so that } \frac{A-B}{2} = 45^{\circ}.$$

$$\text{Also } \frac{A+B}{2} = 60^{\circ} 38' 57''; \text{ whence } A = 105^{\circ} 38' 57'', B = 15^{\circ} 38' 57''.$$

$$7. \quad \sin B = \frac{b \sin A}{a} = \frac{12 \sin 30^{\circ}}{9};$$

$$\therefore \log \sin B = 1.07918 + \bar{1}.69897 - .95424 = \bar{1}.82391;$$

$$\therefore B = 41^{\circ} 48' 39'' \text{ or } 138^{\circ} 11' 21'', \text{ both values being admissible since } a < b.$$

$$\therefore C = 108^{\circ} 11' 21'' \text{ or } 11^{\circ} 48' 39''.$$

$$\text{Again} \quad c = \frac{b \sin C_1}{\sin B_1} = \frac{12 \sin 108^{\circ} 11' 21''}{\sin 41^{\circ} 48' 39''};$$

$$\therefore \log c = 1.07918 + \bar{1}.97774 - \bar{1}.82391 = 1.23301;$$

$$\therefore c = 17.1.$$

$$\text{Similarly from } c = \frac{b \sin C_2}{\sin B_2}, \text{ we easily obtain } c = 3.68.$$

$$8. \quad \tan \frac{C}{2} = \frac{a-b}{a+b} \cot \frac{A-B}{2} = \frac{1}{2} \cot 45^{\circ} = \frac{1}{2};$$

$$\therefore \log \tan \frac{C}{2} = \log 1 - \log 2 = \bar{1}.6989700$$

$$\begin{array}{r} \log \tan 26^{\circ} 33' \\ \text{diff.} \end{array} \quad \begin{array}{r} = \bar{1}.6986847 \\ \underline{2853} \end{array}$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{2853}{3159} \times 60'' = 54.2'';$$

$$\therefore \frac{C}{2} = 26^{\circ} 33' 54.2'', \text{ and } C = 53^{\circ} 7' 48''.$$

Hence $\frac{A+B}{2} = 63^\circ 26' 6''$, and $\frac{A-B}{2} = 45^\circ$;

$$\therefore A = 108^\circ 26' 6'', \quad B = 18^\circ 26' 6''.$$

9. (1) Let $a=1404$, $b=960$, $A=32^\circ 15'$;

then $\sin B = \frac{b \sin A}{a} = \frac{80}{117 \operatorname{cosec} 32^\circ 15'}$;

$$\therefore \log \sin B = 1 + 3 \log 2 - (2 \log 3 + \log 13 + \log \operatorname{cosec} 32^\circ 15') \\ = \bar{1} \cdot 5621316, \text{ on reduction.}$$

$$\therefore B = 21^\circ 23'; \quad \therefore C = 126^\circ 22'.$$

(2) Let $a=1404$, $b=960$, $B=32^\circ 15'$;

then $\sin A = \frac{117}{80 \operatorname{cosec} 32^\circ 15'}$;

$$\therefore \log \sin A = 2 \log 3 + \log 13 - (1 + 3 \log 2 + \log \operatorname{cosec} 32^\circ 15') \\ = \bar{1} \cdot 8923236, \text{ on reduction.}$$

$$\therefore A = 51^\circ 18', \text{ or } 128^\circ 42' \text{ since the solution is ambiguous.}$$

$$\therefore C = 96^\circ 27', \text{ or } 19^\circ 3'.$$

10. We have $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{1 - \frac{c}{b}}{1 + \frac{c}{b}} \cot \frac{A}{2}$

$$= \frac{1 - \cos \phi}{1 + \cos \phi} \cot \frac{A}{2} = \tan^2 \frac{\phi}{2} \cot \frac{A}{2},$$

where $\cos \phi = \frac{c}{b} = \frac{10}{11}$.

Hence $\log \cos \phi = 1 - \log 11 = \bar{1} \cdot 958607$;

$$\therefore \phi = 24^\circ 37' 12''.$$

Again $\log \tan \frac{B-C}{2} = 2 \log \tan \frac{\phi}{2} + \log \cot \frac{A}{2}$

$$= \bar{2} \cdot 677782 + \cdot 495800$$

$$= \bar{1} \cdot 173582.$$

$$\therefore \frac{B-C}{2} = 8^\circ 28' 56 \cdot 5'', \text{ and } \frac{B+C}{2} = 72^\circ 17' 30'';$$

$$\therefore B = 80^\circ 46' 26 \cdot 5'', \quad C = 63^\circ 48' 33 \cdot 5''.$$

$$11. \quad \sin B = \frac{b}{a} \sin A = \frac{1071}{873} \sin 50^\circ;$$

$$\therefore \log \sin B = 3.029789 + \bar{1}.884254 - 2.941014$$

$$= \bar{1}.973029$$

$$\log \sin 70^\circ = \bar{1}.972986$$

$$\text{diff.} \quad 43$$

$$\log \sin 70^\circ 1' = \bar{1}.973032$$

$$\log \sin 70^\circ = \bar{1}.972986$$

$$\text{diff. for } 1' \quad 46$$

$$\text{prop}^l. \text{ increase} = \frac{43}{46} \times 60'' = 56'';$$

$$\therefore B = 70^\circ 0' 56'', \text{ or } 109^\circ 59' 4'', \text{ both values being admissible since } a < b.$$

12. As in Art. 195 we easily obtain

$$c^2 = (a+b)^2 \sin^2 \frac{C}{2} \left\{ 1 + \left(\frac{a-b}{a+b} \right)^2 \cot^2 \frac{C}{2} \right\}.$$

Now let

$$\frac{a-b}{a+b} \cot \frac{C}{2} = \tan \theta \dots\dots\dots(1),$$

then

$$c = (a+b) \sin \frac{C}{2} \sec \theta \dots\dots\dots(2).$$

From (1),

$$\tan \theta = \frac{1}{3} \cot 18^\circ 26' 6'',$$

$$\therefore \log \tan \theta = .47712 - .47712 = 0.$$

Whence

$$\tan \theta = 1, \text{ and } \theta = 45^\circ.$$

From (2),

$$c = 9 \sqrt{2} \sin 18^\circ 26' 6'';$$

$$\therefore \log c = 2 \log 3 + \frac{1}{2} \log 2 + \log \sin 18^\circ 26' 6''$$

$$= .95424 + .15052 + \bar{1}.5$$

$$= .60476.$$

$$\therefore c = 4.0249.$$

$$13. \text{ Here } \sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}, \text{ and } s-a=549, s-c=291;$$

$$\therefore \log \sin \frac{B}{2} = \frac{1}{2} (\log 549 + \log 291 - \log 1000 - \log 1258).$$

$$\log 549 = 2.7395723$$

$$\log 291 = 2.4638930$$

$$5.2034653$$

$$6.0996806$$

$$2) \bar{1}.1037847$$

$$\log \sin \frac{B}{2} = \bar{1}.5518924$$

$$\log 1000 = 3$$

$$\log 1258 = 3.0996806$$

$$6.0996806$$

$$\text{Diff. for } 60'' = 3313;$$

$$\log \sin 20^\circ 52' = \bar{1}.5516871$$

$$\text{diff.} \quad 2053$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{2053}{3313} \times 60'' = 37''.$$

$$\therefore \frac{B}{2} = 20^\circ 52' 37'', \text{ and } B = 41^\circ 45' 14''.$$

$$14. \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{326 \times 199}{976 \times 451}}.$$

$$\log 326 = 2.5132176$$

$$\log 199 = 2.2988531$$

$$4.8120707$$

$$5.6436263$$

$$2) \bar{1}.1684444$$

$$\log \tan \frac{A}{2} = \bar{1}.5842222$$

$$\log \tan 21^{\circ} 0' = \bar{1}.5841774$$

$$\text{diff.} \quad 448$$

$$\log 976 = 2.9894498$$

$$\log 451 = 2.6541765$$

$$5.6436263$$

$$\text{Diff. for } 20'' = 3775;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{448}{3775} \times 60'' = 7'';$$

$$\therefore \frac{A}{2} = 21^{\circ} 0' 7'', \text{ and } A = 42^{\circ} 0' 14''.$$

$$\text{Again } \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{451 \times 199}{976 \times 326}};$$

$$\log 199 = 2.2988531$$

$$\log 451 = 2.6541765$$

$$4.9530296$$

$$5.5026674$$

$$2) \bar{1}.4503622$$

$$\log \tan \frac{B}{2} = \bar{1}.7251811$$

$$\log \tan 27^{\circ} 58' = \bar{1}.7250646$$

$$\text{diff.} \quad 1165$$

$$\log 976 = 2.9894498$$

$$\log 326 = 2.5132176$$

$$5.5026674$$

$$\text{Diff. for } 60'' = 3.049;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{1165}{3049} \times 60'' = 23'';$$

$$\therefore \frac{B}{2} = 27^{\circ} 58' 23'', \text{ and } B = 55^{\circ} 56' 46''.$$

$$\therefore C = 82^{\circ} 3'.$$

$$15. \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{2\frac{1}{2} \times 1\frac{1}{2}}{5 \times 6}} = \sqrt{\frac{1}{8}};$$

$$\therefore \log \sin \frac{A}{2} = -\frac{3}{2} \log 2$$

$$= \bar{1}.5484550,$$

$$\log \sin 20^{\circ} 42' = \bar{1}.5483585$$

$$\text{diff.} \quad 965$$

$$\text{Diff. for } 1' = 3342;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{965}{3342} \times 60'' = 17.3'';$$

$$\therefore A = 41^{\circ} 24' 35''.$$

$$16. \text{ Here } \sin C = \frac{c}{b} = \frac{28.58}{57.321}.$$

$$\begin{array}{r} \log 28.58 = 1.4560622 \\ \log 57.321 = 1.7583138 \\ \hline \log \sin 29^\circ 54' = 1.6977484 \\ \log \sin 29^\circ 54' = 1.6976545 \\ \text{diff.} \qquad \qquad \qquad 939 \end{array}$$

$$\text{Diff. for } 60'' = 2196;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{939}{2196} \times 60'' = 26''.$$

$$\therefore C = 29^\circ 54' 26''; \text{ whence } A = 60^\circ 5' 34''.$$

17. Let C be the right angle, and $A = 18^\circ 37' 29''$; then

$$c = \frac{a}{\sin A} = \frac{284}{\sin 18^\circ 37' 29''}.$$

$$\begin{array}{r} \log 284 \qquad \qquad = 2.4533183 \\ \log \sin 18^\circ 37' 29'' = 1.5042917 \\ \log c \qquad \qquad \qquad = 2.9490266 \\ \log 889.25 \qquad = 2.9490239 \\ \hline \qquad \qquad \qquad \qquad \qquad 27 \\ \qquad \qquad \qquad \qquad \qquad 5 \qquad \qquad \qquad 25 \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad 20 \\ \qquad \qquad \qquad \qquad \qquad 4 \qquad \qquad \qquad 20 \\ \hline \end{array}$$

$$\begin{array}{r} \log \sin 18^\circ 37' = 1.5041105 \\ \frac{29}{60''} \times 3748 = 1812 \\ \hline \log \sin 18^\circ 37' 29'' = 1.5042917 \end{array}$$

$$\therefore c = 889.2554 \text{ feet.}$$

$$18. \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{1}{8} \cdot \sqrt{3};$$

$$\begin{array}{r} \therefore \log \tan \frac{B-C}{2} = \frac{1}{2} \log 3 - 3 \log 2 \\ \qquad \qquad \qquad = 1.3354706 \\ \log \tan 12^\circ 12' = 1.3348711 \\ \text{diff.} \qquad \qquad \qquad 5995 \end{array}$$

$$\text{Diff. for } 60'' = 6112;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{5995}{6112} \times 60'' = 59''.$$

$$\therefore \frac{B-C}{2} = 12^\circ 12' 59'', \text{ and } \frac{B+C}{2} = 60^\circ;$$

$$\therefore B = 72^\circ 12' 59'', \quad C = 47^\circ 47' 1''.$$

19. Let AC be the ladder, C the window, and B the foot of the wall; then from the right-angled triangle ABC ,

$$\begin{aligned} AC=b &= \frac{42\cdot37}{\sin 72^{\circ} 15'}, \\ \log b &= 1\cdot6270585 - \bar{1}\cdot9788175 \\ &= 1\cdot6482410 \\ \log 44\cdot487 &= \frac{1\cdot6482331}{79} \\ &\quad 8 \quad \quad \quad 78 \\ \therefore \text{length of ladder} &= 44\cdot4878 \text{ feet.} \end{aligned}$$

$$\begin{aligned} 20. \quad \tan \frac{A-B}{2} &= \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{9\cdot99}{53\cdot91} \cot 17^{\circ} 30'. \\ \log 9\cdot99 &= \cdot9995655 \\ \log \cot 17^{\circ} 30' &= \cdot5012777 \\ &\quad 1\cdot5008432 \\ \log 53\cdot91 &= 1\cdot7316693 \\ &\quad 1\cdot7691739 \\ \log \tan 30^{\circ} 26' &= 1\cdot7689922 \\ \text{diff.} &\quad 1817 \end{aligned}$$

$$\text{Diff. for } 60'' = 2892;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{1817}{2892} \times 60'' = 38'';$$

$$\therefore \frac{A-B}{2} = 30^{\circ} 26' 38'', \text{ and } \frac{A+B}{2} = 72^{\circ} 30';$$

$$\therefore A = 102^{\circ} 56' 38'', B = 42^{\circ} 3' 22''.$$

$$\begin{aligned} 21. \quad \tan \frac{B-C}{2} &= \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{11\cdot29}{38\cdot95} \cot 23^{\circ} 37' 30''. \\ \log \cot 23^{\circ} 37' &= \cdot3592844 \\ \text{Subtract } \frac{30}{60} \times 3441 &\quad 1721 \\ &\quad \cdot3591123 \\ \log 11\cdot29 &= 1\cdot0526939 \\ &\quad 1\cdot4118062 \\ \log 38\cdot95 &= 1\cdot5905075 \\ \log \tan \frac{B-C}{2} &= 1\cdot8212987 \\ \log \tan 33^{\circ} 31' &= 1\cdot8210574 \\ \text{diff.} &\quad 2413 \end{aligned}$$

$$\text{Diff. for } 60'' = 2743;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{2413}{2743} \times 60'' = 53''.$$

$$\therefore \frac{B-C}{2} = 33^{\circ} 31' 53'', \text{ and } \frac{B+C}{2} = 66^{\circ} 22' 30'';$$

$$\therefore B = 99^{\circ} 54' 23'', C = 32^{\circ} 50' 37''.$$

Again $a = \frac{b \sin A}{\sin B} = \frac{25.12 \sin 47^\circ 15'}{\sin 99^\circ 54' 23''} = \frac{25.12 \sin 47^\circ 15'}{\cos 9^\circ 54' 23''}.$

$$\log \sin 47^\circ 15' = \bar{1}.8658868$$

$$\log 25.12 = 1.4000196$$

$$\underline{1.2659064}$$

$$\bar{1}.9934760$$

$$\log a = \underline{1.2724304}$$

$$\log 18.725 = \underline{1.2724218}$$

$$86$$

$$4 \quad \underline{93}$$

$$\log \cos 9^\circ 54' = \bar{1}.9934844$$

$$\text{Subtract } \frac{23}{60} \times 220 = \underline{84}$$

$$\bar{1}.9934760$$

$$\therefore a = 18.7254.$$

22. $\sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}} = \sqrt{\frac{1361.12 \times 1024.48}{1837.2 \times 2173.84}}.$

$$\log 1361.1 = 3.1338900$$

$$2 \quad \underline{64}$$

$$\log 1024.4 = 3.0104696$$

$$8 \quad \underline{339}$$

$$6.1443999$$

$$\underline{6.6013840}$$

$$2) \bar{1}.5430159$$

$$\log \sin \frac{B}{2} = \bar{1}.7715079$$

$$\log \sin 36^\circ 13' = \bar{1}.7714702$$

$$\text{diff.} \quad \underline{377}$$

$$\text{Diff. for } 60'' = 1724;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{377}{1724} \times 60'' = 13''.$$

$$\therefore \frac{B}{2} = 36^\circ 13' 13'', \text{ and } B = 72^\circ 26' 26''.$$

23. $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{6.4405 \times 14.9114}{52.1248 \times 30.7728}}.$

$$\log 6.4405 = .8089196$$

$$\log 14.911 = 1.1735068$$

$$4 \quad \underline{116}$$

$$\bar{1}.9824380$$

$$\underline{3.2052113}$$

$$2) \bar{2}.7772267$$

$$\log \tan \frac{A}{2} = \bar{1}.3886134$$

$$\log \tan 13^\circ 44' = \bar{1}.3880837$$

$$\underline{5297}$$

$$\text{Diff. for } 60'' = 5475;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{5297}{5475} \times 60'' = 58''.$$

$$\therefore \frac{A}{2} = 13^\circ 44' 58'', \text{ and } A = 27^\circ 29' 56''.$$

Again $\tan \frac{B}{2} = \sqrt{\frac{14.9114 \times 30.7728}{52.1248 \times 6.4405}}.$

$$\begin{array}{rcl} \log 14.9114 & = & 1.1735184 \\ \log 30.7728 & = & 1.4881670 \\ & & 2.6616854 \\ & & 2.5259639 \\ & 2) & \underline{.1357215} \end{array}$$

$$\begin{array}{rcl} \log \tan \frac{B}{2} & = & .0678608 \\ \log \tan 49^{\circ} 27' & = & .0677338 \\ \text{diff.} & & \underline{1270} \end{array}$$

$$\begin{array}{rcl} \log 52.1248 & = & 1.7170443 \\ \log 6.4405 & = & .8089196 \\ & & \underline{2.5259639} \end{array}$$

Diff. for $60'' = 2558$;

$\therefore$ prop^l. increase = $\frac{1270}{2558} \times 60'' = 30''.$

$\therefore \frac{B}{2} = 49^{\circ} 27' 30'',$ and $B = 98^{\circ} 55'.$

$\therefore C = 53^{\circ} 35' 4''.$

24. $\tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{202.949}{1497.597} \cot 51^{\circ} 36' 27''.$

$$\begin{array}{rcl} \log \cot 51^{\circ} 36' & = & \bar{1}.8990487 \\ \text{Subtract } \frac{27}{60} \times 2595 & = & \underline{1168} \\ & & \bar{1}.8989319 \\ \log 202.94 & = & 2.3073677 \\ 9 & & \underline{193} \\ & & 2.2063189 \\ & & \underline{3.1753949} \end{array}$$

$$\begin{array}{rcl} \log 1497.5 & = & 3.1753668 \\ 9 & & 261 \\ 7 & & \underline{203} \\ & & 3.1753949 \end{array}$$

$$\begin{array}{rcl} \log \tan \frac{C-B}{2} & = & \bar{1}.0309240 \\ \log \tan 6^{\circ} 7' & = & \bar{1}.0300464 \\ \text{diff.} & & \underline{8776} \end{array}$$

Diff. for $60'' = 11909$; $\therefore$ prop^l. increase = $\frac{8776}{11909} \times 60'' = 44''$;

$\therefore \frac{C-B}{2} = 6^{\circ} 7' 44'',$ and $\frac{C+B}{2} = 38^{\circ} 23' 33'';$

$\therefore C = 44^{\circ} 31' 17'', B = 32^{\circ} 15' 49''.$

To find a , we have $a = \frac{b \sin A}{\sin B} = \frac{647.324 \sin 103^\circ 12' 54''}{\sin 32^\circ 15' 49''}$.

Now $\log \sin 103^\circ 12' 54'' = \log \sin 76^\circ 47' 6''$.

$$\log \sin 76^\circ 47' = \bar{1}.9883415$$

$$\frac{6}{60} \times 297 = \frac{30}{\bar{1}.9883445}$$

$$\log 647.32 = 2.8111190$$

$$4 \quad 27$$

$$2.7994662$$

$$\bar{1}.7273911$$

$$\log a = 3.0720751$$

$$\log 1180.5 = 3.0720660$$

$$91$$

$$2 \quad 73$$

$$180$$

$$5 \quad 185$$

$$\log \sin 32^\circ 15' = \bar{1}.7272276$$

$$\frac{49}{60} \times 2002 = \frac{1635}{\bar{1}.7273911}$$

$$\therefore a = 1180.525.$$

25.

$$a = \frac{b \sin A}{\sin B} = \frac{23.2783 \sin 37^\circ 57'}{\sin 43^\circ 13'}.$$

$$\log 23.278 = 1.3669457$$

$$3 \quad 56$$

$$\log \sin 37^\circ 57' = \bar{1}.7888565$$

$$\bar{1}.1558078$$

$$\log \sin 43^\circ 13' = \bar{1}.8355378$$

$$\log a = 1.3202700$$

$$\log 20.905 = 1.3202502$$

$$198$$

$$9 \quad 187$$

$$\therefore a = 20.9059.$$

Again
$$c = \frac{b \sin C}{\sin B} = \frac{23 \cdot 2783 \sin 81^\circ 10'}{\sin 43^\circ 13'}.$$

$$\begin{aligned} \log c &= 1 \cdot 3669503 + \bar{1} \cdot 9948181 - \bar{1} \cdot 8355378 \\ &= 1 \cdot 5262316 \\ \log 33 \cdot 591 &= 1 \cdot 5262229 \\ &\quad \underline{87} \\ &\quad 7 \quad \quad 90 \\ &\quad \quad \quad \underline{} \\ \therefore c &= 33 \cdot 5917. \end{aligned}$$

26.
$$b = \frac{c \sin B}{\sin C} = \frac{2484 \cdot 3 \sin 72^\circ 43' 25''}{\sin 47^\circ 12' 17''}.$$

$\begin{aligned} \log \sin 72^\circ 43' &= \bar{1} \cdot 9799339 \\ \frac{25}{60} \times 393 &= 164 \\ \log 2484 \cdot 3 &= 3 \cdot 3952040 \\ &\quad \underline{3 \cdot 3751543} \\ &\quad \bar{1} \cdot 8655693 \\ \log b &= 3 \cdot 5095850 \\ \log 3232 \cdot 8 &= 3 \cdot 5095788 \\ &\quad \underline{62} \\ &\quad 4 \quad \quad 54 \\ &\quad 6 \quad \quad \underline{8} \end{aligned}$	$\begin{aligned} \log \sin 47^\circ 12' &= \bar{1} \cdot 8655362 \\ \frac{17}{60} \times 1169 &= 331 \\ &\quad \underline{\bar{1} \cdot 8655693} \end{aligned}$
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$$\therefore b = 3232 \cdot 846.$$

Again
$$a = \frac{c \sin A}{\sin C} = \frac{2484 \cdot 3 \sin 60^\circ 4' 18''}{\sin 47^\circ 12' 17''}.$$

$$\begin{aligned} \log \sin 60^\circ 4' &= \bar{1} \cdot 9378220 \\ \frac{18}{60} \times 727 &= 218 \\ \log 2484 \cdot 3 &= 3 \cdot 3952040 \\ &\quad \underline{3 \cdot 3330478} \\ \log \sin 47^\circ 12' 17'' &= \bar{1} \cdot 8655693 \\ &\quad \underline{3 \cdot 4674785} \\ \log 2934 \cdot 1 &= 3 \cdot 4674749 \\ &\quad \underline{36} \\ &\quad 2 \quad \quad 30 \\ &\quad 4 \quad \quad \underline{6} \end{aligned}$$

$$\therefore a = 2934 \cdot 124.$$

$$27. \tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{4367}{4667} \cot 15^\circ 45'.$$

$$\log \tan \frac{C-B}{2} = 3.6401832 + .5497060 - 3.6690378$$

$$= .5208514$$

$$\log \tan 73^\circ 13' = \frac{.5205681}{2833}$$

$$\text{Diff. for } 60'' = 4568;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{2833}{4568} \times 60'' = 37''.$$

$$\therefore \frac{C-B}{2} = 73^\circ 13' 37'', \text{ and } \frac{C+B}{2} = 74^\circ 15';$$

$$\therefore C = 147^\circ 28' 37'', B = 1^\circ 1' 23''.$$

Again

$$a = \frac{c \sin A}{\sin C} = \frac{4517 \sin 31^\circ 30'}{\sin 32^\circ 31' 23''}.$$

$$\log 4517 = 3.6548501$$

$$\log \sin 31^\circ 30' = 1.7180851$$

$$3.3729352$$

$$\log \sin 32^\circ 31' 23'' = 1.7304907$$

$$\log a = 3.6424445$$

$$\log 4389.8 = 3.6424447$$

$$\log \sin 32^\circ 31' = 1.7304148$$

$$\frac{23}{60} \times 1981 = 759$$

$$\log \sin 32^\circ 31' 23'' = 1.7304907$$

$$\therefore a = 4389.8 \text{ nearly.}$$

$$28. \sin A = \frac{a \sin C}{c} = \frac{324.68 \sin 35^\circ 17' 12''}{421.73}.$$

$$\log \sin 35^\circ 17' = 1.7616424$$

$$\frac{12}{60} \times 1784 = 357$$

$$\log 324.68 = 2.5114555$$

$$2.2731336$$

$$\log 421.73 = 2.6250345$$

$$\log \sin A = 1.6480991$$

$$\log \sin 26^\circ 24' = 1.6480038$$

$$\text{diff. } 953$$

$$\text{Diff. for } 60'' = 2544;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{953}{2544} \times 60'' = 23'';$$

$$\therefore A = 26^\circ 24' 23'', \text{ and } \therefore B = 118^\circ 18' 25''.$$

Again

$$b = \frac{c \sin B}{\sin C} = \frac{421.73 \sin 61^\circ 41' 35''}{\sin 35^\circ 17' 12''}.$$

$$\log \sin 61^\circ 41' = \bar{1}.9446501$$

$$\frac{35}{60} \times 680 = 397$$

$$\log 421.73 = 2.6250345$$

$$2.5697243$$

$$\log \sin 35^\circ 17' 12'' = \bar{1}.7616781$$

$$2.8080462$$

$$\log 642.75 = 2.8080421$$

$$41$$

$$6 \quad 41$$

$$\therefore b = 642.756.$$

$$29. \quad \sin C = \frac{c \sin A}{a} = \frac{435.6 \sin 36^\circ 18' 27''}{321.7}.$$

$$\log \sin 36^\circ 18' = \bar{1}.7723314$$

$$\frac{27}{60} \times 1719 = 774$$

$$\log 435.6 = 2.6390879$$

$$2.4114967$$

$$\log 321.7 = 2.5074511$$

$$\log \sin C = \bar{1}.9040456$$

$$\log \sin 53^\circ 17' = \bar{1}.9039587$$

$$\text{diff.} \quad 869$$

$$\text{Diff. for } 60'' = 943;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{869}{943} \times 60'' = 55'';$$

$\therefore C = 53^\circ 17' 55''$, or $126^\circ 42' 5''$, both values being admissible since $a < c$.

$$30. \quad \sin C = \frac{c \sin B}{b} = \frac{1665}{1325} \sin 52^\circ 19'.$$

$$\log \sin C = 3.2214142 + \bar{1}.8983968 - 3.1222159$$

$$= \bar{1}.9975951$$

$$\log \sin 83^\circ 58' = \bar{1}.9975877$$

$$\text{diff.} \quad 74$$

$$\text{Diff. for } 60'' = 134;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{74}{134} \times 60'' = 33''.$$

$\therefore C = 83^\circ 58' 33''$, or $96^\circ 1' 27''$, both values being admissible since $b < c$.

Now, with diagram of page 132, we have $\angle BAC_2 = 31^\circ 39' 33''$.

$$\therefore a = \frac{b \sin A}{\sin B} = \frac{1325 \sin 31^\circ 39' 33''}{\sin 52^\circ 19'}$$

$$\log \sin 31^\circ 39' = \bar{1}.7199350$$

$$\frac{33}{60} \times 2049 = 1127$$

$$\log 1325 = 3.1222159$$

$$\underline{2.8422636}$$

$$\log \sin 52^\circ 19' = \bar{1}.8983968$$

$$\log a = 2.9438668$$

$$\log 878.75 = 2.9438653$$

$$\begin{array}{r} 15 \\ 3 \quad 15 \\ \hline \end{array}$$

$$\therefore a = 878.753.$$

$$31. \text{ Here } A = 64^\circ 26' 15'', \quad b = \frac{a \sin B}{\sin A} = \frac{3795 \sin 73^\circ 15' 15''}{\sin 64^\circ 26' 15''}.$$

$$\log \sin 73^\circ 15' = \bar{1}.9811711$$

$$\frac{15}{60} \times 380 = 95$$

$$\log 3795 = 3.5792118$$

$$3.5603924$$

$$\log \sin 64^\circ 26' 15'' = \bar{1}.9552620$$

$$\log 6 = 3.6051304$$

$$\log 4028.3 = 3.6051218$$

$$\begin{array}{r} 86 \\ 8 \quad 86 \\ \hline \end{array}$$

$$\log \sin 64^\circ 26' = \bar{1}.9552469$$

$$\frac{15}{60} \times 604 = 151$$

$$\underline{\bar{1}.9552620}$$

$$\therefore b = 4028.38.$$

Again

$$c = \frac{a \sin C}{\sin A} = \frac{3795 \sin 42^\circ 18' 30''}{\sin 64^\circ 26' 15''}.$$

$$\log 3795 = 3.5792118$$

$$\log \sin 42^\circ 18' = \bar{1}.8280231$$

$$\frac{30}{60} \times 1388 = 694$$

$$\underline{3.4073043}$$

$$\log \sin A = \bar{1}.9552620$$

$$\log c = 3.4520423$$

$$\log 2831.6 = 3.4520319$$

$$\begin{array}{r} 104 \\ 7 \quad 107 \\ \hline \end{array}$$

$$\therefore c = 2831.67.$$

$$32. \quad \sin B = \frac{b}{c} \sin C = \frac{17}{12} \sin 43^\circ 12' 12''.$$

$$\log 17 = 1.2304489$$

$$\log \sin 43^\circ 12' = 1.8354033$$

$$\frac{12}{60} \times 1345 = 269$$

$$\hline 1.0658791$$

$$\log 12 = 1.0791812$$

$$\log \sin B = 1.9866979$$

$$\log \sin 75^\circ 53' = 1.9866827$$

$$\text{diff.} \quad \hline 152$$

$$\text{Diff. for } 60'' = 317;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{152}{317} \times 60'' = 29''.$$

$$\therefore B = 75^\circ 53' 29'', \text{ or } 104^\circ 6' 31'', \text{ both values being admissible since } c < b.$$

$$\therefore A = 60^\circ 54' 19'', \text{ or } 32^\circ 41' 17''.$$

$$33. \quad \text{Let } b = 2.7402, c = .7401, A = 59^\circ 27' 5''.$$

$$\tan \frac{B-C}{2} = \frac{2.0001}{3.4803} \cot 29^\circ 43' 32.5''.$$

$$\log \cot 29^\circ 43' = .2435347$$

$$\frac{32.5}{60} \times 2934 = 1589$$

$$\log 2.0001 = .3010517$$

$$\hline .5444275$$

$$\log 3.4803 = .5416167$$

$$\log \tan \frac{B-C}{2} = .0028108$$

$$\log \tan 45^\circ 11' = .0027793$$

$$\text{diff.} \quad \hline 315$$

$$\text{Diff. for } 60'' = 2527;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{315}{2527} \times 60'' = 7.5''.$$

$$\therefore \frac{B-C}{2} = 45^\circ 11' 7.5'', \text{ and } \frac{B+C}{2} = 60^\circ 16' 27.5'';$$

$$\therefore B = 105^\circ 27' 35'', C = 15^\circ 5' 20''.$$

Again

$$a = \frac{c \sin A}{\sin C} = \frac{.7401 \sin 59^\circ 27' 5''}{\sin 15^\circ 5' 20''}.$$

$$\log \sin 59^\circ 27' = 1.9350969$$

$$\frac{5}{60} \times 746 = 62$$

$$\log .7401 = 1.8692904$$

$$\hline 1.8043935$$

$$\hline 1.4155029$$

$$\log a = .3888906$$

$$\log 2.4484 = .3888824$$

$$\hline 82$$

$$5 \quad \hline 89$$

$$\log \sin 15^\circ 5' = 1.4153468$$

$$\frac{20}{60} \times 4684 = 1561$$

$$\hline 1.4155029$$

$$\therefore a = 2.44845.$$

Let h = the altitude; then $h = b \sin C$;

$$\begin{aligned}\therefore \log h &= \log 2.7402 + \log \sin 15^\circ 5' 20'' \\ &= .4379823 + \bar{1}.4155029 \\ &= \bar{1}.8532852 \\ \log .71332 &= \bar{1}.8532844 \\ &\quad \begin{array}{r} 8 \\ 1 \quad 6 \\ \hline \end{array} \\ \therefore \text{altitude} &= .713321.\end{aligned}$$

34. Let $b = 105.25$, $c = 76.75$, $B - C = 17^\circ 48''$;

then
$$\cot \frac{A}{2} = \frac{b+c}{b-c} \tan \frac{B-C}{2} = \frac{182}{28.5} \tan 8^\circ 54'.$$

$$\begin{aligned}\log \cot \frac{A}{2} &= \log 182 + \log \tan 8^\circ 54' - \log 28.5 \\ &= 2.2600714 + \bar{1}.1947802 - 1.4548449 \\ &= .0000067 = \log \cot 45^\circ \text{ nearly.} \\ \therefore A &= 90^\circ \text{ nearly.}\end{aligned}$$

35. (1) $\sin C = \frac{c \sin A}{a} = \frac{36.5 \sin 43^\circ 15'}{20};$

$$\begin{aligned}\log \sin C &= 1.5622929 + \bar{1}.8358066 - 1.3010300 \\ &= .0970695,\end{aligned}$$

which is impossible, since $\sin C$ must be < 1 .

(2) $\sin C = \frac{36.5 \sin 43^\circ 15'}{30};$

$$\begin{aligned}\log \sin 43^\circ 15' &= \bar{1}.8358066 \\ \log 36.5 &= 1.5622929 \\ &\quad \bar{1}.3980995 \\ \log 30 &= 1.4771213 \\ \log \sin C &= \bar{1}.9209782\end{aligned}$$

Thus C is not a right angle, and since $a < c$ the solution is ambiguous.

(3) $\sin C = \frac{36.5 \sin 43^\circ 15'}{45}.$

$$\begin{aligned}\log \sin C &= 1.3980995 - 1.6532125 \\ &= \bar{1}.7448870\end{aligned}$$

Diff. for $60'' = 1890$;

$$\begin{aligned}\log \sin 33^\circ 45' &= \bar{1}.7447390 \\ \text{diff.} &\quad \underline{1480}\end{aligned}$$

$$\therefore \text{prop}^1. \text{ increase} = \frac{148}{189} \times 60'' = 47''.$$

$$\therefore C = 33^\circ 45' 47''; \therefore B = 102^\circ 59' 13''.$$

$$\text{Now } b = \frac{a \sin B}{\sin A} = \frac{45 \cos 12^\circ 59' 13''}{\sin 43^\circ 15'};$$

$$\log \cos 12^\circ 59' = \bar{1}.9887531$$

$$\text{Subtract } \frac{13}{60} \times 292 = \quad 63$$

$$\bar{1}.9887468$$

$$\log 45 = \bar{1}.6532125$$

$$\bar{1}.6419593$$

$$\log \sin 43^\circ 15' = \bar{1}.8358066$$

$$\bar{1}.8061527$$

$$\log 63.996 = \bar{1}.8061528$$

$$\text{Thus } b = 63.996.$$

36. For the first part of the Example, see Art. 197.

$$\tan \theta = \frac{2 \sqrt{17.32 \times 13.47}}{3.85} \sin 23^\circ 36' 30'';$$

$$\log \sin 23^\circ 36' = \bar{1}.6024388$$

$$\frac{30}{60} \times 2890 = \quad 1445$$

$$\log 2 = \quad .3010300$$

$$\bar{1}.1839578$$

$$\bar{1}.0875711$$

$$\log 3.85 = \quad .5854607$$

$$\log \tan \theta = \quad .5021104$$

$$\log \tan 72^\circ 31' = \quad .5017184$$

$$3920$$

$$\therefore \theta = 72^\circ 31' 53''.$$

$$\log 17.32 = 1.2385479$$

$$\log 13.47 = 1.1293676$$

$$2 \quad 2.3679155$$

$$\bar{1}.1839578$$

$$\text{Diff. for } 60'' = 4410;$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{392}{441} \times 60'' = 53''.$$

$$\text{Again } c = (a - b) \sec \theta;$$

$$\log 3.85 = \quad .5854607$$

$$\log \sec 72^\circ 31' = \quad .5222591$$

$$\frac{53}{60} \times 4013 = \quad 3545$$

$$\bar{1}.1080743$$

$$\log 12.825 = \bar{1}.1080574$$

$$169$$

$$5 \quad \bar{1}.170$$

$$\therefore c = 12.8255.$$

37. See Art. 195.

$$\begin{aligned} \tan \phi &= \frac{44.1}{10.5} \tan 22^\circ 36'; \\ \log 44.1 &= 1.6444386 \\ \log \tan 22^\circ 36' &= 1.6193645 \\ &\quad \underline{1.2638031} \\ \log 10.5 &= 1.0211893 \\ \log \tan \phi &= .2426138 & \text{Diff. for } 60'' = 2930; \\ \log \tan 60^\circ 13' &= .2423617 & \therefore \text{prop}^l. \text{ increase} = \frac{2521}{2930} \times 60'' = 52''. \\ \text{diff.} &\quad \underline{2521} \end{aligned}$$

$$\therefore \phi = 60^\circ 13' 52''.$$

$$\text{Again } c = (a - b) \cos \frac{C}{2} \sec \phi;$$

$$\begin{aligned} \log 10.5 &= 1.0211893 \\ \log \cos 22^\circ 36' &= 1.9653006 \\ \log \sec 60^\circ 13' &= .3038870 \\ &\quad \frac{52}{60} \times 2208 = \quad \underline{1914} \\ \log c &= 1.2905683 \\ \log 19.523 &= 1.2905466 \\ &\quad \underline{217} \\ &\quad 9 \quad \underline{200} \\ &\quad \quad \underline{170} \\ &\quad 8 \quad \underline{178} \\ \therefore c &= 19.52398. \end{aligned}$$

EXAMPLES. XVI. g. PAGE 183 D.

$$\begin{aligned} 1. \quad \sin \frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{16.3 \times 9}{11.7 \times 19}}. \\ \log 16.3 &= 1.2122 & \log 11.7 &= 1.0682 \\ \log 9 &= .9542 & \log 19 &= 1.2787 \\ &\quad \underline{2.1664} & &\quad \underline{2.3469} \\ &\quad 2.3469 \\ 2) &\underline{1.8195} \\ \log \sin \frac{A}{2} &= 1.9097; \text{ whence } \frac{A}{2} = 54^\circ 19'. \\ \therefore A &= 108^\circ 38'. \end{aligned}$$

$$2. \quad \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}} = \sqrt{\frac{112.5 \times 19.25}{68.75 \times 63}}.$$

$\log 112.5 = 2.0511$ $\log 19.25 = 1.2844$ $\hline 3.3355$ 3.6366 $2) \bar{1}.6989$		$\log 68.75 = 1.8373$ $\log 63 = 1.7993$ $\hline 3.6366$
--	--	--

$$\log \cos \frac{B}{2} = \bar{1}.8495; \text{ whence } \frac{B}{2} = 45^\circ.$$

$$\therefore B = 90^\circ.$$

$$3. \quad \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}} = \sqrt{\frac{5.72 \times 3.31}{11.24 \times 13.65}}.$$

$\log 5.72 = .7572$ $\log 3.31 = .5198$ $\hline 1.2772$ 2.1859 $2) \bar{1}.0913$		$\log 11.24 = 1.0508$ $\log 13.65 = 1.1351$ $\hline 2.1859$
--	--	---

$$\log \sin \frac{C}{2} = \bar{1}.5457; \text{ whence } \frac{C}{2} = 20^\circ 34'.$$

$$\therefore C = 41^\circ 8'.$$

$$4. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{1 \times 14}{23 \times 8}}.$$

$\log 14 = 1.1461$ 2.2648 $2) \bar{2}.8813$		$\log 23 = 1.3617$ $\log 8 = .9031$ $\hline 2.2648$
---	--	---

$$\log \tan \frac{A}{2} = \bar{1}.4407; \text{ whence } \frac{A}{2} = 15^\circ 25'. \quad \therefore A = 30^\circ 50'.$$

$$\text{Again,} \quad \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{8 \times 14}{23 \times 1}}.$$

$\log 8 = .9031$ $\log 14 = 1.1461$ $\hline 2.0492$ 1.3617 $2) \bar{0}.6875$		$\log 23 = 1.3617$
--	--	--------------------

$\log \tan \frac{B}{2} = .3438$ $\log \tan 65^\circ 36' = .3433$ $\text{diff.} \quad \hline 5$	Since 5 is the mean of the differences 3 and 7, the corresponding increase in the angle is $1^\circ 5'$.
--	--

$$\text{Hence } \frac{B}{2} = 65^\circ 37'.5, \text{ and } B = 131^\circ 15'.$$

$$\therefore C = 17^\circ 55'.$$

$$5. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{24.22 \times 7.32}{70.72 \times 39.18}}.$$

$\log 24.22 = 1.3842$ $\log 7.32 = .8645$ 2.2487 3.4426 $2) \overline{2.8061}$	$\log 70.72 = 1.8495$ $\log 39.18 = 1.5931$ 3.4426
--	---

$$\tan \frac{A}{2} = \bar{1}.4031; \text{ whence } \frac{A}{2} = 14^\circ 12'. \quad \therefore A = 28^\circ 24'.$$

Again, $\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{39.18 \times 7.32}{70.72 \times 24.22}}.$

$\log 39.18 = 1.5931$ $\log 7.32 = .8645$ 2.4576 3.2337 $2) \overline{1.2239}$	$\log 70.72 = 1.8495$ $\log 24.22 = 1.3842$ 3.2337
--	---

$$\log \tan \frac{B}{2} = \bar{1}.6120; \text{ whence } \frac{B}{2} = 22^\circ 15'.$$

$$\therefore B = 44^\circ 30', \text{ and } C = 107^\circ 6'.$$

$$6. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{11.02 \times 22.38}{82.62 \times 49.22}}.$$

$\log 11.02 = 1.0422$ $\log 22.38 = 1.3498$ 2.3920 3.6093 $2) \overline{2.7827}$	$\log 82.62 = 1.9171$ $\log 49.22 = 1.6922$ 3.6093
--	---

$$\tan \frac{A}{2} = \bar{1}.3914; \text{ whence } \frac{A}{2} = 13^\circ 50'. \quad \therefore A = 27^\circ 40'.$$

Again, $\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{49.22 \times 22.38}{82.62 \times 11.02}}.$

$\log 49.22 = 1.6922$ $\log 22.38 = 1.3498$ 3.0420 2.9593 $2) \overline{0.0827}$	$\log 82.62 = 1.9171$ $\log 11.02 = 1.0422$ 2.9593
--	---

$$\tan \frac{B}{2} = .0414; \text{ whence } \frac{B}{2} = 47^\circ 43'.5, \text{ as in Ex. 4 above.}$$

$$\therefore B = 95^\circ 27'; \text{ and } C = 56^\circ 53'.$$

$$7. \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{15.98 \times 1.23}{6.84 \times 10.37}}.$$

$\log 15.98 = 1.2036$ $\log 1.23 = .0899$ 1.2935 1.8508 $2) \ 1.4427$	$\log 6.84 = .8351$ $\log 10.37 = 1.0157$ 1.8508
---	---

$$\log \cos \frac{A}{2} = 1.7214; \text{ whence } \frac{A}{2} = 58^\circ 14'. \quad \therefore A = 116^\circ 28'.$$

$$8. \quad \sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}} = \sqrt{\frac{10.85 \times 2.85}{21.5 \times 29.5}}.$$

$\log 10.85 = 1.0355$ $\log 2.85 = .4548$ 1.4903 2.8022 $2) \ 2.6881$	$\log 21.5 = 1.3324$ $\log 29.5 = 1.4698$ 2.8022
---	---

$$\log \sin \frac{B}{2} = 1.3441; \text{ whence } \frac{B}{2} = 12^\circ 45' .5, \text{ as in Ex. 4 above.}$$

$$\therefore B = 25^\circ 31'.$$

$$9. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$= \sqrt{\frac{3 \times 2}{10 \times 5}}$ $= \sqrt{.12}.$	$= \sqrt{\frac{5 \times 2}{10 \times 3}}$ $= \sqrt{\frac{1}{3}}.$
--	--

$$\therefore \log \tan \frac{A}{2} = \frac{1}{2} (\bar{1}.0792)$$

$$= \bar{1}.5396;$$

$$\text{whence } \frac{A}{2} = 19^\circ 6'.$$

$$\therefore \log \tan \frac{B}{2} = \frac{1}{2} (-\log 3) = \frac{1}{2} (\bar{1}.5229)$$

$$= \bar{1}.7614;$$

$$\text{whence } \frac{B}{2} = 30^\circ.$$

Thus $A = 38^\circ 12'$, $B = 60^\circ$, $C = 81^\circ 48'$.

$$10. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{7 \times 6}{21 \times 8}}.$$

$\log 7 = .8451$ $\log 6 = .7782$ 1.6233 2.2253 $2) \ 1.3980$	$\log 21 = 1.3222$ $\log 8 = .9031$ 2.2253
---	---

$$\tan \frac{A}{2} = 1.6990; \text{ whence } \frac{A}{2} = 26^\circ 34'. \quad \therefore A = 53^\circ 8'.$$

$$\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{8 \times 6}{21 \times 7}}$$

$$\log 8 = \cdot 9031$$

$$\log 6 = \cdot 7782$$

$$\underline{1\cdot6813}$$

$$2\cdot1673$$

$$2) \underline{1\cdot5140}$$

$$\log 21 = 1\cdot3222$$

$$\log 7 = \cdot 8451$$

$$\underline{2\cdot1673}$$

$$\log \tan \frac{B}{2} = \bar{1}\cdot7570; \text{ whence } \frac{B}{2} = 29^\circ 45'.$$

$$\therefore B = 59^\circ 30', \text{ and } C = 67^\circ 22'.$$

$$11. \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{17\cdot8}{47\cdot8} \cot 53^\circ 43'.$$

$$\log 17\cdot8 = 1\cdot2504$$

$$\log \cot 53^\circ 43' = \bar{1}\cdot8658$$

$$\underline{1\cdot1162}$$

$$\log 47\cdot8 = \underline{1\cdot6794}$$

$$\log \tan \frac{B-C}{2} = \bar{1}\cdot4368;$$

$$\frac{B+C}{2} = 36^\circ 17',$$

$$\text{whence } \frac{B-C}{2} = 15^\circ 18'.$$

$$\therefore B = 51^\circ 35', \text{ and } C = 20^\circ 59'.$$

$$12. \quad \tan \frac{B-A}{2} = \frac{b-a}{b+a} \cot \frac{C}{2} = \frac{38\cdot7}{232\cdot1} \cot 61^\circ 51'.$$

$$\log 38\cdot7 = 1\cdot5877$$

$$\log \cot 61^\circ 51' = \bar{1}\cdot7284$$

$$\underline{1\cdot3161}$$

$$\log 232\cdot1 = \underline{2\cdot3657}$$

$$\log \tan \frac{B-A}{2} = \bar{2}\cdot9504;$$

$$\frac{B+A}{2} = 28^\circ 9',$$

$$\text{whence } \frac{B-A}{2} = 5^\circ 6'.$$

$$\therefore B = 33^\circ 15', \text{ and } A = 23^\circ 3'.$$

$$13. \quad \tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2} = \frac{564}{588} \cot 30^\circ 15'.$$

$$\log 564 = 2\cdot7513$$

$$\log \cot 30^\circ 15' = \cdot 2342$$

$$\underline{2\cdot9855}$$

$$\log 588 = \underline{2\cdot7694}$$

$$\log \tan \frac{C-A}{2} = \cdot 2161;$$

$$\frac{C+A}{2} = 59^\circ 45',$$

$$\text{whence } \frac{C-A}{2} = 58^\circ 42'.$$

$$\therefore C = 118^\circ 27', \text{ and } A = 1^\circ 3'.$$

14. Here $a=27.3$, $b=16.8$, $C=45^\circ 7'$. Required A , B , and c .

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{10.5}{44.1} \cot 22^\circ 33'.5.$$

$$\log 10.5 = 1.0212$$

$$\log \cot 22^\circ 33'.5 = .3815$$

$$\hline 1.4027$$

$$\log 44.1 = 1.6444$$

$$\log \tan \frac{A-B}{2} = \overline{1.7583};$$

$$\frac{A+B}{2} = 67^\circ 26'.5,$$

$$\text{whence } \frac{A-B}{2} = 29^\circ 49'.$$

$$\therefore A = 97^\circ 15'.5, \text{ and } B = 37^\circ 37'.5.$$

Again,

$$c = \frac{b \sin C}{\sin B} = \frac{16.8 \sin 45^\circ 7'}{\sin 37^\circ 37'.5}.$$

$$\log 16.8 = 1.2253$$

$$\log \sin 45^\circ 7' = \overline{1.8503}$$

$$\hline 1.0756$$

$$\log \sin 37^\circ 37'.5 = \overline{1.7857}$$

$$\log c = \overline{1.2899}; \text{ whence } c = 19.49.$$

- 15.

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{57.8}{108.0} \cot 18^\circ 30'.$$

$$\log 57.8 = 1.7619$$

$$\log \cot 18^\circ 30' = .4755$$

$$\hline 2.2374$$

$$\log 108 = 2.0334$$

$$\log \tan \frac{B-C}{2} = .2040;$$

$$\frac{B+C}{2} = 71^\circ 30',$$

$$\text{whence } \frac{B-C}{2} = 57^\circ 59'.$$

$$\therefore B = 129^\circ 29', \text{ and } C = 13^\circ 31'.$$

Again,

$$a = \frac{c \sin A}{\sin C} = \frac{25.1 \sin 37^\circ}{\sin 13^\circ 31'}.$$

$$\log 25.1 = 1.3997$$

$$\log \sin 37^\circ = \overline{1.7795}$$

$$\hline 1.1792$$

$$\log \sin 13^\circ 31' = \overline{1.3687}$$

$$\log a = \overline{1.8105}; \text{ whence } a = 64.65.$$

- 16.

$$\tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{6.48}{60.48} \cot 30^\circ.$$

$$\log 6.48 = .8116$$

$$\log \cot 30^\circ = .2386$$

$$\hline 1.0502$$

$$\log 60.48 = \overline{1.7816}$$

$$\log \tan \frac{C-B}{2} = \overline{1.2686};$$

$$\frac{C+B}{2} = 60^\circ,$$

$$\text{whence } \frac{C-B}{2} = 10^\circ 31'.$$

$$\therefore C = 70^\circ 31', \text{ and } B = 49^\circ 29'.$$

$$17. \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{8}{22} \cot 41^{\circ} 7' = \frac{4}{11} \cot 41^{\circ} 7'.$$

$$\log 4 = \cdot 6021$$

$$\log \cot 41^{\circ} 7' = \cdot 0590$$

$$\hline \cdot 6611$$

$$\log 11 = \underline{1 \cdot 0414}$$

$$\log \tan \frac{A-B}{2} = \underline{1 \cdot 6197};$$

$$\frac{A+B}{2} = 48^{\circ} 53',$$

$$\text{whence } \frac{A-B}{2} = 22^{\circ} 37'.$$

$$\therefore A = 71^{\circ} 30', \text{ and } B = 26^{\circ} 16'.$$

$$18. \quad \text{Let } a=2b, \text{ and } C=52^{\circ} 47', \text{ then}$$

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{1}{3} \cot 26^{\circ} 23' \cdot 5$$

$$= \frac{1}{3} \times 2 \cdot 0152 = \cdot 6717 = \tan 33^{\circ} 53'.$$

$$\therefore \frac{A+B}{2} = 63^{\circ} 36' \cdot 5, \text{ and } \frac{A-B}{2} = 33^{\circ} 53'.$$

$$\therefore A = 97^{\circ} 29' \cdot 5, \text{ and } B = 29^{\circ} 43' \cdot 5.$$

$$19. \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{44}{218} \cot 9^{\circ} 8'.$$

$$\log 44 = 1 \cdot 6435$$

$$\log \cot 9^{\circ} 8' = \cdot 7938$$

$$\hline 2 \cdot 4373$$

$$\log 218 = \underline{2 \cdot 3385}$$

$$\log \tan \frac{B-C}{2} = \cdot 0988;$$

$$\frac{B+C}{2} = 80^{\circ} 52',$$

$$\text{whence } \frac{B-C}{2} = 51^{\circ} 28'.$$

$$\therefore B = 132^{\circ} 20', \text{ and } C = 29^{\circ} 24'.$$

$$20. \quad \tan \frac{B-A}{2} = \frac{b-a}{b+a} \cot \frac{C}{2} = \frac{6 \cdot 43}{41 \cdot 63} \cot 60^{\circ} 49'.$$

$$\log 6 \cdot 43 = \cdot 8082$$

$$\log \cot 60^{\circ} 49' = \underline{1 \cdot 7470}$$

$$\hline \cdot 5552$$

$$\log 41 \cdot 63 = \underline{1 \cdot 6194}$$

$$\log \tan \frac{B-A}{2} = \underline{2 \cdot 9358};$$

$$\frac{B+A}{2} = 29^{\circ} 11',$$

$$\text{whence } \frac{B-A}{2} = 4^{\circ} 56'.$$

$$\therefore B = 34^{\circ} 7', \text{ and } A = 24^{\circ} 15'.$$

$$\text{Again, } c = \frac{a \sin C}{\sin A} = \frac{17 \cdot 6 \sin 121^{\circ} 38'}{\sin 24^{\circ} 15'} = \frac{17 \cdot 6 \sin 58^{\circ} 22'}{\sin 24^{\circ} 15'}.$$

$$\log 17 \cdot 6 = 1 \cdot 2455$$

$$\log \sin 58^{\circ} 22' = \underline{1 \cdot 9301}$$

$$\hline 1 \cdot 1755$$

$$\log \sin 24^{\circ} 15' = \underline{1 \cdot 6135}$$

$$\log c = \underline{1 \cdot 5620}; \text{ whence } c = 36 \cdot 48.$$

$$\begin{aligned}
 21. \quad a &= \frac{b \sin A}{\sin B} = \frac{100 \sin 40^\circ}{\sin 70^\circ} . \\
 \log 100 &= 2 \\
 \log \sin 40^\circ &= \overline{1.8081} \\
 &\quad \underline{1.8081} \\
 \log \sin 70^\circ &= \overline{1.9730} \\
 \log a &= 1.8351; \text{ whence } a = 68.41.
 \end{aligned}$$

$$\begin{aligned}
 22. \quad b &= \frac{a \sin B}{\sin A} = \frac{85.2 \sin 42^\circ}{\sin 31^\circ} . \\
 \log 85.2 &= 1.9304 \\
 \log \sin 42^\circ &= \overline{1.8255} \\
 &\quad \underline{1.7559} \\
 \log \sin 31^\circ &= \overline{1.7118} \\
 \log b &= 2.0441; \text{ whence } b = 110.7.
 \end{aligned}$$

$$\begin{aligned}
 23. \quad a &= \frac{c \sin A}{\sin C} = \frac{5.23 \sin 49^\circ 11'}{\sin 109^\circ 34'} = \frac{5.23 \sin 49^\circ 11'}{\sin 70^\circ 26'} . \\
 \log 5.23 &= .7185 \\
 \log \sin 49^\circ 11' &= \overline{1.8789} \\
 &\quad \underline{.5974} \\
 \log \sin 70^\circ 26' &= \overline{1.9742} \\
 \log a &= .6232; \text{ whence } a = 4.200.
 \end{aligned}$$

$$\begin{aligned}
 24. \quad c &= \frac{b \sin C}{\sin B} = \frac{873 \sin 71^\circ 35'}{\sin 42^\circ 58'} . \\
 \log 873 &= 2.9410 \\
 \log \sin 71^\circ 35' &= \overline{1.9772} \\
 &\quad \underline{2.9182} \\
 \log \sin 42^\circ 58' &= \overline{1.8336} \\
 \log c &= 3.0846; \text{ whence } c = 1215.
 \end{aligned}$$

$$\begin{aligned}
 25. \quad a &= \frac{c \sin A}{\sin C} = \frac{60 \sin 60^\circ}{\sin 40^\circ 40'} . \\
 \log 60 &= 1.7782 \\
 \log \sin 60^\circ &= \overline{1.9375} \\
 &\quad \underline{1.7157} \\
 \log \sin 40^\circ 40' &= \overline{1.8140} \\
 \log a &= 1.9017; \text{ whence } a = 79.75.
 \end{aligned}$$

$$\begin{aligned}
 26. \quad a &= \frac{c \sin A}{\sin C} = \frac{3.57 \sin 51^\circ 51'}{\sin 40^\circ 26'}. \\
 \log 3.57 &= .5527 \\
 \log \sin 51^\circ 51' &= \overline{1.8956} \\
 &\quad .4483 \\
 \log \sin 40^\circ 26' &= \overline{1.8120} \\
 \log a &= .6363 \\
 \therefore a &= 4.328.
 \end{aligned}$$

$$\begin{aligned}
 b &= \frac{c \sin B}{\sin C} = \frac{3.57 \sin 87^\circ 43'}{\sin 40^\circ 26'}. \\
 \log 3.57 &= .5527 \\
 \log \sin 87^\circ 43' &= \overline{1.9996} \\
 &\quad .5523 \\
 \log \sin 40^\circ 26' &= \overline{1.8120} \\
 \log b &= .7403 \\
 \therefore b &= 5.499.
 \end{aligned}$$

$$\begin{aligned}
 27. \quad b &= \frac{a \sin B}{\sin A} = \frac{125.7 \sin 65^\circ 47'}{\sin 61^\circ 34'}. \\
 \log 125.7 &= 2.0993 \\
 \log \sin 65^\circ 47' &= \overline{1.9600} \\
 &\quad 2.0593 \\
 \log \sin 61^\circ 34' &= \overline{1.9442} \\
 \log b &= 2.1151; \text{ whence } b = 130.3.
 \end{aligned}$$

$$\begin{aligned}
 28. \quad a &= \frac{c \sin A}{\sin C} \\
 &= \frac{92.93 \sin 72^\circ 19'}{\sin 24^\circ 24'}. \\
 \log 92.93 &= 1.9681 \\
 \log \sin 72^\circ 19' &= \overline{1.9789} \\
 &\quad 1.9470 \\
 \log \sin 24^\circ 24' &= \overline{1.6161} \\
 \log a &= 2.3309 \\
 \therefore a &= 214.2.
 \end{aligned}$$

$$\begin{aligned}
 b &= \frac{c \sin B}{\sin C} \\
 &= \frac{92.93 \sin 83^\circ 17'}{\sin 24^\circ 24'}. \\
 \log 92.93 &= 1.9681 \\
 \log \sin 83^\circ 17' &= \overline{1.9970} \\
 &\quad 1.9651 \\
 \log \sin 24^\circ 24' &= \overline{1.6161} \\
 \log b &= 2.3490 \\
 \therefore b &= 223.4.
 \end{aligned}$$

$$\begin{aligned}
 29. \quad b &= \frac{a \sin B}{\sin A} \\
 &= \frac{4.375 \sin 49^\circ 30'}{\sin 60^\circ}. \\
 \log 4.375 &= .6410 \\
 \log \sin 49^\circ 30' &= \overline{1.8810} \\
 &\quad .5220 \\
 \log \sin 60^\circ &= \overline{1.9375} \\
 \log b &= .5845 \\
 \therefore b &= 3.841.
 \end{aligned}$$

$$\begin{aligned}
 c &= \frac{a \sin C}{\sin A} \\
 &= \frac{4.375 \sin 70^\circ 30'}{\sin 60^\circ}. \\
 \log 4.375 &= .6410 \\
 \log \sin 70^\circ 30' &= \overline{1.9743} \\
 &\quad .6153 \\
 \log \sin 60^\circ &= \overline{1.9375} \\
 \log c &= .6778 \\
 \therefore c &= 4.762.
 \end{aligned}$$

$$30. \quad \frac{A}{1} = \frac{B}{4} = \frac{C}{7} = \frac{A+B+C}{12} = \frac{180^\circ}{12} = 15^\circ.$$

$$\therefore A = 15^\circ, B = 60^\circ, C = 105^\circ.$$

$$a = \frac{b \sin A}{\sin B} = \frac{89.36 \sin 15^\circ}{\sin 60^\circ}.$$

$$\log 89.36 = 1.9512$$

$$\log \sin 15^\circ = \bar{1}.4130$$

$$\underline{1.3642}$$

$$\log \sin 60^\circ = \bar{1}.9375$$

$$\log a = \bar{1}.4267$$

$$\therefore a = 26.71.$$

$$c = \frac{b \sin C}{\sin B} = \frac{89.36 \sin 75^\circ}{\sin 60^\circ}.$$

$$\log 89.36 = 1.9512$$

$$\log \sin 75^\circ = \bar{1}.9849$$

$$\underline{1.9361}$$

$$\log \sin 60^\circ = \bar{1}.9375$$

$$\log c = \bar{1}.9986$$

$$\therefore c = 99.68.$$

$$31. \quad \sin B = \frac{b \sin A}{a} = \frac{62 \sin 82^\circ 14'}{73}.$$

$$\log 62 = 1.7924$$

$$\log \sin 82^\circ 14' = \bar{1}.9960$$

$$\underline{1.7884}$$

$$\log 73 = 1.8633$$

$$\log \sin B = \bar{1}.9251; \text{ whence } B = 57^\circ 18'.$$

$$32. \quad \sin C = \frac{c \sin B}{b} = \frac{63.45 \sin 27^\circ 15'}{41.62}.$$

$$\log 63.45 = 1.8024$$

$$\log \sin 27^\circ 15' = \bar{1}.6607$$

$$\underline{1.4631}$$

$$\log 41.62 = 1.6193$$

$$\log \sin C = \bar{1}.8438;$$

whence $C = 44^\circ 16'$, or $135^\circ 44'$, both values being admissible since $b < c$.

$$33. \quad \sin A = \frac{a \sin B}{b} = \frac{17.28 \sin 55^\circ 13'}{23.97}.$$

$$\log 17.28 = 1.2375$$

$$\log \sin 55^\circ 13' = \bar{1}.9145$$

$$\underline{1.1520}$$

$$\log 23.97 = 1.3797$$

$$\log \sin A = \bar{1}.7723$$

$$\therefore A = 36^\circ 18'$$

$$\text{and } C = 88^\circ 29'.$$

$$c = \frac{a \sin C}{\sin A} = \frac{17.28 \sin 88^\circ 29'}{\sin 36^\circ 18'}.$$

$$\log 17.28 = 1.2375$$

$$\log \sin 88^\circ 29' = \bar{1}.9999$$

$$\underline{1.2374}$$

$$\log \sin 36^\circ 18' = \bar{1}.7723$$

$$\log c = 1.4651$$

$$\therefore c = 29.18.$$

$$34. \quad \sin B = \frac{b \sin A}{a} = \frac{141.3 \sin 40^\circ}{94.2}.$$

$\log \sin B = 2.1501 + \bar{1}.8081 - 1.9741 = \bar{1}.9841$; whence $B = 74^\circ 36'$, or $105^\circ 24'$, since $a < b$. $\therefore C_1 = 65^\circ 24'$ and $C_2 = 34^\circ 36'$.

$$c_1 = \frac{a \sin C_1}{\sin A} = \frac{94.2 \sin 65^\circ 24'}{\sin 40^\circ}.$$

$$\log 94.2 = 1.9741$$

$$\log \sin 65^\circ 24' = \bar{1}.9587$$

$$\underline{1.9328}$$

$$\log \sin 40^\circ = \bar{1}.8081$$

$$\log c_1 = 2.1247$$

$$\therefore c_1 = 133.2.$$

$$c_2 = \frac{a \sin C_2}{\sin A} = \frac{94.2 \sin 34^\circ 36'}{\sin 40^\circ}.$$

$$\log 94.2 = 1.9741$$

$$\log \sin 34^\circ 36' = \bar{1}.7542$$

$$\underline{1.7283}$$

$$\log \sin 40^\circ = \bar{1}.8081$$

$$\log c_2 = 1.9202$$

$$\therefore c_2 = 83.22.$$

$$35. \quad \sin B = \frac{b \sin A}{a} = \frac{137 \sin 20^\circ 41'}{115}.$$

$\therefore \log \sin B = 2.1367 + \bar{1}.5481 - 2.0607 = \bar{1}.6241$; whence $B = 24^\circ 53'$, or $155^\circ 7'$, since $a < b$. $\therefore C_1 = 134^\circ 26'$ and $C_2 = 4^\circ 12'$.

$$c_1 = \frac{a \sin C_1}{\sin A} = \frac{115 \sin 45^\circ 34'}{\sin 20^\circ 41'}.$$

$$\log 115 = 2.0607$$

$$\log \sin 45^\circ 34' = \bar{1}.8538$$

$$\underline{1.9145}$$

$$\log \sin 20^\circ 41' = \bar{1}.5481$$

$$\log c_1 = 2.3664$$

$$\therefore c_1 = 232.5.$$

$$c_2 = \frac{a \sin C_2}{\sin A} = \frac{115 \sin 4^\circ 12'}{\sin 20^\circ 41'}.$$

$$\log 115 = 2.0607$$

$$\log \sin 4^\circ 12' = \bar{2}.8647$$

$$\underline{.9254}$$

$$\log \sin 20^\circ 41' = \bar{1}.5481$$

$$\log c_2 = \bar{1}.3773$$

$$\therefore c_2 = 23.84.$$

$$36. \quad \sin C = \frac{c \sin B}{b} = \frac{1665}{1325} \sin 52^\circ 19'.$$

$$\log 1665 = 3.2214$$

$$\log \sin 52^\circ 19' = \bar{1}.8984$$

$$\underline{3.1198}$$

$$\log 1325 = 3.1222$$

$$\log \sin C = \bar{1}.9976$$

$\therefore C = 84^\circ$, or 96° , both values being admissible since $b < c$.

Now, with diagram of page 132, we have $\angle BAC_2 = 31^\circ 41'$;

$$\therefore a = \frac{b \sin A}{\sin B} = \frac{1325 \sin 31^\circ 41'}{\sin 52^\circ 19'}.$$

$$\log 1325 = 3.1222$$

$$\log \sin 31^\circ 41' = \bar{1}.7203$$

$$\underline{2.8425}$$

$$\log \sin 52^\circ 19' = \bar{1}.8984$$

$$\log a = 2.9441; \text{ whence } a = 879.2.$$

$$\begin{array}{lcl}
 37. & \sin A = \frac{a \sin C}{c} = \frac{324.7 \sin 35^\circ}{421.7} & \\
 & \log 324.7 = 2.5114 & \\
 & \log \sin 35^\circ = 1.7586 & \\
 & \quad 2.2700 & \\
 & \log 421.7 = 2.6250 & \\
 & \log \sin A = 1.6450 & \\
 & \therefore A = 26^\circ 12', & \\
 & \text{and } B = 118^\circ 48'. & \\
 & b = \frac{c \sin B}{\sin C} = \frac{421.7 \sin 61^\circ 12'}{35^\circ} & \\
 & \log 421.7 = 2.6250 & \\
 & \log \sin 61^\circ 12' = 1.9427 & \\
 & \quad 2.5677 & \\
 & \log \sin 35^\circ = 1.7586 & \\
 & \log b = 2.8091 & \\
 & \therefore b = 644.3. &
 \end{array}$$

$$\begin{array}{lcl}
 38. & \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2} = \frac{9.99}{53.91} \cot 17^\circ 30'. & \\
 & \log 9.99 = .9996 & \\
 & \log \cot 17^\circ 30' = .5013 & \\
 & \quad 1.5009 & \\
 & \log 53.91 = 1.7317 & \\
 & \log \tan \frac{A-B}{2} = 1.7692; & \text{whence } \frac{A-B}{2} = 30^\circ 27'. \\
 & & \therefore A = 102^\circ 57', \text{ and } B = 42^\circ 3'.
 \end{array}$$

$$\begin{array}{lcl}
 39. & \sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}} = \sqrt{\frac{549 \times 291}{1000 \times 1258}} & \\
 & \log 549 = 2.7396 & \log 1000 = 3 \\
 & \log 291 = 2.4639 & \log 1258 = 3.0997 \\
 & \quad 5.2035 & \quad 6.0997 \\
 & \quad 6.0997 & \\
 & 2) 1.1038 & \\
 & \log \sin \frac{B}{2} = 1.5519; & \text{whence } \frac{B}{2} = 20^\circ 52'. \\
 & & \therefore B = 41^\circ 44'.
 \end{array}$$

$$\begin{array}{lcl}
 40. & \sin B = \frac{b \sin C}{c} = \frac{17 \sin 43^\circ 12'}{12} & \\
 & \log 17 = 1.2304 & \\
 & \log \sin 43^\circ 12' = 1.8354 & \\
 & \quad 1.0658 & \\
 & \log 12 = 1.0792 & \\
 & \log \sin B = 1.9866 & \\
 & & \therefore B = 75^\circ 51', \text{ or } 104^\circ 9', \\
 & & \text{both values being admissible} \\
 & & \text{since } c < b. \\
 & & \therefore A = 60^\circ 57', \text{ or } 32^\circ 39'.
 \end{array}$$

$$41. \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{326 \times 199}{976 \times 451}}.$$

log 326 =	2.5132		log 976 =	2.9894
log 199 =	2.2989		log 451 =	2.6542
	4.8121			5.6436
	5.6436			
	2) 1.1685			

$$\log \tan \frac{A}{2} = 1.5843; \text{ whence } \frac{A}{2} = 21^\circ$$

$$\therefore A = 42^\circ.$$

Again,

$$\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{451 \times 199}{976 \times 326}}.$$

log 451 =	2.6542		log 976 =	2.9894
log 199 =	2.2989		log 326 =	2.5132
	4.9531			5.5026
	5.5026			
	2) 1.4505			

$$\log \tan \frac{B}{2} = 1.7253; \text{ whence } \frac{B}{2} = 27^\circ 59'.$$

$$\therefore B = 55^\circ 58', \text{ and } C = 82^\circ 2'.$$

$$42. \quad \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{2.5 \times 1.5}{5 \times 6}} = \sqrt{\frac{1}{8}}.$$

$$\therefore \log \sin \frac{A}{2} = -\frac{3}{2} \log 2 = 1.5485.$$

$$\therefore \frac{A}{2} = 20^\circ 42', \text{ and } A = 41^\circ 24'.$$

$$43. \quad \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{11.29}{38.95} \cot 23^\circ 37'.5.$$

$$\log \cot 23^\circ 36' = .3596$$

subtract for $1'.5 = \frac{5}{38.95}$ (the mean of diffs. 3 and 7)

$$\log 11.29 = 1.0527$$

$$1.4118$$

$$\log 38.95 = 1.5905$$

$$\log \tan \frac{B-C}{2} = 1.8213;$$

$$\frac{B+C}{2} = 66^\circ 22'.5,$$

$$\text{whence } \frac{B-C}{2} = 33^\circ 32'.$$

$$\therefore B = 99^\circ 54'.5, \text{ and } C = 32^\circ 50'.5.$$

$$\text{Again, } a = \frac{b \sin A}{\sin B} = \frac{25 \cdot 12 \sin 47^\circ 15'}{\sin 99^\circ 54' \cdot 5} = \frac{25 \cdot 12 \sin 47^\circ 15'}{\sin 80^\circ 5' \cdot 5}.$$

$$\log 25 \cdot 12 = 1 \cdot 4000$$

$$\log \sin 47^\circ 15' = \overline{1} \cdot 8658$$

$$\overline{1} \cdot 2658$$

$$\log \sin 80^\circ 5' \cdot 5 = \overline{1} \cdot 9935$$

$$\log a = \overline{1} \cdot 2723; \text{ whence } a = 18 \cdot 72.$$

$$44. \quad \tan \frac{C-B}{2} = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{4367}{4667} \cot 15^\circ 45'.$$

$$\log 4367 = 3 \cdot 6402$$

$$\log \cot 15^\circ 45' = \cdot 5498$$

$$\overline{4} \cdot 1900$$

$$\log 4667 = 3 \cdot 6691$$

$$\frac{C+B}{2} = 74^\circ 15',$$

$$\log \tan \frac{C-B}{2} = \cdot 5209; \text{ whence } \frac{C-B}{2} = 73^\circ 14'.$$

$$\therefore C = 147^\circ 29', \text{ and } B = 1^\circ 1'.$$

$$\text{Again, } a = \frac{c \sin A}{\sin C} = \frac{4517 \sin 31^\circ 30'}{\sin 147^\circ 29'} = \frac{4517 \sin 31^\circ 30'}{\sin 32^\circ 31'}.$$

$$\log 4517 = 3 \cdot 6549$$

$$\log \sin 31^\circ 30' = \overline{1} \cdot 7181$$

$$\overline{3} \cdot 3730$$

$$\log \sin 32^\circ 31' = \overline{1} \cdot 7304$$

$$\log a = 3 \cdot 6426; \text{ whence } a = 4391.$$

$$45. \quad \sin C = \frac{c \sin A}{a} = \frac{435 \cdot 6 \sin 36^\circ 18'}{321 \cdot 7}.$$

$$\log 435 \cdot 6 = 2 \cdot 6391$$

$$\log 36^\circ 18' = \overline{1} \cdot 7723$$

$$\overline{2} \cdot 4114$$

$$\log 321 \cdot 7 = 2 \cdot 5074$$

$$\log \sin C = \overline{1} \cdot 9040;$$

whence $C = 53^\circ 17'$, or $126^\circ 43'$, both values being admissible since $a < c$.

46. For the first part of the example, see Art. 197.

$$\tan \theta = \frac{2\sqrt{17 \cdot 32 \times 13 \cdot 47}}{3 \cdot 85} \sin 23^\circ 36'.$$

$$\log 17 \cdot 32 = 1 \cdot 2385$$

$$\log 13 \cdot 47 = 1 \cdot 1294$$

$$2) \overline{2} \cdot 3679$$

$$\overline{1} \cdot 1840$$

$$1 \cdot 0874$$

$$\log 2 = \cdot 3010$$

$$\log 3 \cdot 85 = \cdot 5855$$

$$\log \sin 23^\circ 36' = \overline{1} \cdot 6024$$

$$\log \tan \theta = \cdot 5019$$

$$\overline{1} \cdot 0874;$$

$$\text{whence } \theta = 72^\circ 31', \text{ approx.}$$

Again,
$$c = \frac{a-b}{\cos \theta} = \frac{3.85}{\cos 72^\circ 31'}.$$

$$\begin{aligned} \log 3.85 &= .5855 \\ \log \cos 72^\circ 31' &= \overline{1.4777} \\ \log c &= 1.1078; \text{ whence } c = 12.81. \end{aligned}$$

47. See Art. 195.

$$\begin{aligned} \tan \phi &= \frac{44.1}{10.5} \tan 22^\circ 36'. \\ \log 44.1 &= 1.6444 \\ \log \tan 22^\circ 36' &= \overline{1.6194} \\ &1.2638 \\ \log 10.5 &= \overline{1.0212} \\ \log \tan \phi &= .2426; \text{ whence } \phi = 60^\circ 14', \text{ approx.} \end{aligned}$$

Again,
$$c = \frac{(a-b) \cos \frac{C}{2}}{\cos \phi} = \frac{10.5 \cos 22^\circ 36'}{\cos 60^\circ 14'}.$$

$$\begin{aligned} \log 10.5 &= 1.0212 \\ \log \cos 22^\circ 36' &= \overline{1.9653} \\ &.9865 \\ \log \cos 60^\circ 14' &= \overline{1.6959} \\ \log c &= 1.2906; \text{ whence } c = 19.53. \end{aligned}$$

EXAMPLES. XVII. a. PAGE 185.

1. See figure on page 184.

Let PC represent the cliff, and A and B the two objects. Then $PC = 200$ ft.; $\angle PAC = 30^\circ$, $\angle PBC = 45^\circ$.

$$AB = \frac{BP \sin APB}{\sin PAB} = \frac{BP \sin 15^\circ}{\sin 30^\circ};$$

and
$$BP = \frac{PC}{\sin PBC} = \frac{200}{\sin 45^\circ};$$

$$\therefore AB = \frac{200 \sin 15^\circ}{\sin 45^\circ \sin 30^\circ} = 200 (\sqrt{3} - 1) = 146.4 \text{ ft.}$$

2. See figure on page 185.

Let P represent the mountain top, and A, B the two positions of the observer.

Then $\angle PAC = 15^\circ$, $\angle PBC = 75^\circ$, $AB = 1$ mile.

Let x be the height of mountain in feet;

then $x = PB \sin 75^\circ$; and $PB = \frac{AB \sin 15^\circ}{\sin 60^\circ}$;

$$\begin{aligned}\therefore x &= \frac{1760 \cdot 3 \cdot \sin 15^\circ \sin 75^\circ}{\sin 60^\circ} = \frac{880 \cdot 3 \cdot (\cos 60^\circ - \cos 90^\circ)}{\sin 60^\circ} \\ &= \frac{880 \cdot 3}{\sqrt{3}} = 880\sqrt{3} = 1524 \text{ ft.}\end{aligned}$$

3. Let A, B be the position of the two forts, P the first position of the ship, and Q its position after moving 4 miles towards A ;

then $PQ = 4$ miles, $\angle QPB = 30^\circ$, $\angle AQB = 48^\circ$; $\therefore \angle QBP = 18^\circ$.

$$\therefore QB = \frac{QP \sin 30^\circ}{\sin 18^\circ} = \frac{8}{\sqrt{5}-1} = 2(\sqrt{5}+1) = 6.472 \text{ miles.}$$

4. See figure on page 184.

Let PC represent the tower and A, B the two objects; then

$PC = h$, $\angle PAC = 45^\circ - A$, $\angle PBC = 45^\circ + A$, $\angle APB = 2A$;

$$\therefore AB = \frac{PB \sin 2A}{\sin (45^\circ - A)}, \text{ and } PB = \frac{h}{\sin (45^\circ + A)};$$

$$\therefore AB = \frac{2h \sin 2A}{2 \sin (45^\circ - A) \sin (45^\circ + A)} = \frac{2h \sin 2A}{\cos 2A - \cos 90^\circ} = 2h \tan 2A.$$

5. In the figure on page 184, take D in AB , so that $CD = CP$; then

$AD = a$ feet, $DB = b$ feet, $PC = x$ feet, suppose.

Since $\angle CDP = 45^\circ = \angle DPC$, $\therefore DC = PC = x$.

Also $AC = x + a$, $BC = x - b$, $\angle BPC = A$.

Now from $\triangle PAC$, $\tan A = \frac{x}{x+a} = \frac{x-b}{x}$, in $\triangle BPC$.

$$\therefore x^2 = (x+a)(x-b); \text{ whence } x = \frac{ab}{a-b}.$$

6. Let P, Q be the two positions of the observer.

Then from the figure, since

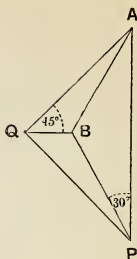
$$\angle AQP = 90^\circ, \quad \angle QPA = 45^\circ;$$

$$\therefore QA = QP = 1 \text{ mile.}$$

And $\triangle$'s ABQ, PBQ are equal in all respects
(Euc. I. 4);

$$\therefore \angle ABQ = \angle PBQ = 120^\circ;$$

$$\therefore AB = \frac{AQ \sin 45^\circ}{\sin 120^\circ} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{1}{3}\sqrt{6} = .816 \text{ miles.}$$



7. Let A be the base, and B the top of the tower, and let C be the point of observation 40 feet up the hill.

Then $AC = 40$ ft., $\angle BAC = 90^\circ - 9^\circ = 81^\circ$, $\angle BCA = 54^\circ$; $\therefore \angle ABC = 45^\circ$;

$$\therefore AB = \frac{AC \sin 54^\circ}{\sin 45^\circ} = 10\sqrt{2}(\sqrt{5} + 1) = 45.76 \text{ feet.}$$

8. See figure on page 186.

Let PA represent the tower, and PB the flagstaff.

Then $PB = c$ feet, $\angle PCA = \alpha$, $\angle PCB = \beta$, and we have

$$x = CP \sin \alpha = \frac{c \cos (\alpha + \beta) \sin \alpha}{\sin \beta}.$$

9. See figure on page 186.

Let BP represent the flagstaff, and PA the wall, C the point of observation.

Let $\angle BCP = \alpha$, $\angle PCA = \theta$, $CA = a$,

Then $\tan \alpha = .5$, $BP = 20$ ft., $PA = 10$ ft.

$$\text{Now} \quad \tan (\theta + \alpha) = \frac{30}{a}, \quad \tan \theta = \frac{10}{a};$$

$$\therefore 3 \tan \theta = \tan (\theta + \alpha) = \frac{2 \tan \theta + 1}{2 - \tan \theta};$$

$$\therefore 3 \tan^2 \theta - 4 \tan \theta + 1 = 0; \text{ whence } \tan \theta = 1 \text{ or } \frac{1}{3}.$$

10. See figure on page 186.

Let BP represent the statue, PA the tower, and C the point of observation.

Let $\angle PCA = \alpha$, $\angle BCP = \beta$, $BP = x$ feet.

Then $PA = 25$ ft., $CA = 60$ ft.;

$$\therefore \tan \alpha = \frac{25}{60} = \frac{5}{12}, \text{ and } \tan \beta = .125 = \frac{1}{8}.$$

$$\text{Now } x + 25 = 60 \tan (\alpha + \beta) = 60 \left(\frac{\frac{1}{8} + \frac{5}{12}}{1 - \frac{5}{96}} \right) = 34\frac{2}{7};$$

$$\therefore \text{height of statue} = 9\frac{2}{7} \text{ feet.}$$

11. See figure and example on page 187.

Here we have $BC = 9$ ft., $BD = 289$ ft., $BE = 324$ ft.;

$$\therefore \tan (\alpha + \theta) = \frac{324}{x}; \quad \tan \alpha = \frac{289}{x}; \quad \tan \theta = \frac{9}{x}.$$

But

$$\tan (\alpha + \theta) = \frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta};$$

$$\therefore \frac{324}{x} = \frac{\frac{289}{x} + \frac{9}{x}}{1 - \frac{289}{x} \cdot \frac{9}{x}} = \frac{298}{x} \cdot \frac{x^2}{x^2 - 289 \times 9};$$

$$\therefore 324x^2 - 324 \times 289 \times 9 = 298x^2;$$

$$\therefore 26x^2 = 324 \times 289 \times 9 = 18^2 \times 17^2 \times 3^2;$$

$$\therefore x^2 = \frac{18^2 \times 51^2}{26} = 18^2 \times \left(\frac{50}{5}\right)^2 \text{ nearly};$$

thus $x = 180$ ft. nearly.

12. See figure on page 187.

Let BD represent the column, DE the statue, BC the man standing by the column, A the point on the opposite bank of the river.

Then $BC = 6$ ft., $BD = 192$ ft., $BE = 216$ ft.

Let $AB = x$ ft., $\angle EAD = \angle CAB = \theta$, $\angle DAB = \alpha$.

$$\text{Then } \tan (\alpha + \theta) = \frac{216}{x}; \quad \tan \alpha = \frac{192}{x}; \quad \tan \theta = \frac{6}{x};$$

$$\therefore \frac{216}{x} = \frac{\frac{192}{x} + \frac{6}{x}}{1 - \frac{6}{x} \times \frac{192}{x}} = \frac{198}{x} \cdot \frac{x^2}{x^2 - 6 \times 192}.$$

From this equation we obtain $x = 48\sqrt{6}$;

$\therefore$ breadth of river $= 48\sqrt{6} = 117.6$ feet nearly.

13. We have at once from a figure,

$$\tan \alpha = \frac{a}{x}, \quad \tan \beta = \frac{b}{x}, \quad \tan \gamma = \frac{c}{x}.$$

Now

$$a + \beta + \gamma = 180^\circ;$$

$$\therefore \tan a + \tan \beta + \tan \gamma = \tan a \tan \beta \tan \gamma;$$

$$\therefore \frac{a}{x} + \frac{b}{x} + \frac{c}{x} = \frac{abc}{x^3};$$

that is,

$$(a + b + c) x^2 = abc.$$

14. See figure on page 188.

Let P be the top of the hill, A and B the points of observation;

then

$$\angle PAC = 45^\circ, \angle BAC = 15^\circ, \angle PDC = 75^\circ;$$

$$\therefore \angle BPA = 75^\circ - 45^\circ = 30^\circ = \angle PAB;$$

$$\therefore PA = 2AB \cos 30^\circ = 500\sqrt{3} \text{ yards};$$

$$\therefore \text{height of hill} = PC = PA \sin 45^\circ = 250\sqrt{6} \text{ yards} \\ = 750\sqrt{6} \text{ feet.}$$

15. We have $\angle CBA = 30^\circ, \angle BCA = 135^\circ; \therefore \angle BAC = 15^\circ;$

$$\therefore AB = \frac{BC \sin 135^\circ}{\sin 15^\circ} = 1760 \times 3 \times \frac{2}{\sqrt{3} - 1} \text{ ft.} \\ = 1760 \times 3 (\sqrt{3} + 1) \text{ feet};$$

$$\therefore \text{height of mountain} = AB \sin 60^\circ = 880 \times 3 (3 + \sqrt{3}) = 12492 \text{ ft.}$$

16. See figure on page 188.

Let A, B be the two points of observation and P the top of the hill; then in the figure $\angle PAC = a, \angle PDC = \gamma, AB = c$ ft.;

$$\therefore \angle APB = \gamma - a, \angle ABP = \pi - (\gamma - \beta),$$

and

$$AP = \frac{AB \sin (\gamma - \beta)}{\sin (\gamma - a)};$$

$$\therefore \text{height of hill} = AP \sin a \\ = c \sin a \sin (\gamma - \beta) \operatorname{cosec} (\gamma - a) \text{ feet.}$$

17. In the figure let P be the top of the mountain, and A, B the two points of observation.

Then $AE = EB = 800$ ft.;

$$\angle BAE = 15^\circ, \angle BED = 30^\circ.$$

Also $\angle PDC = 75^\circ, \angle PAC = 60^\circ;$

$$\therefore \angle APD = 15^\circ.$$

From $\triangle ABE$ we have

$$AB = 2AE \cos 15^\circ = 1600 \cos 15^\circ \text{ ft.};$$

$$\text{and from } \triangle APB, AP = \frac{AB \sin 120^\circ}{\sin 15^\circ}$$

$$= 800\sqrt{3} \cot 15^\circ = 800\sqrt{3} (2 + \sqrt{3}) \text{ ft.};$$

$$\therefore \text{height of mountain} = AP \sin 60^\circ \\ = 400 \times 3 (2 + \sqrt{3}) = 4478 \text{ ft.} \\ \text{approximately.}$$



EXAMPLES. XVII. b. PAGE 190.

1. In the figure of page 186 let $BP=25$ ft., $PA=15$ ft., $CA=x$ feet.

Then
$$\frac{BC}{x} = \frac{25}{15} = \frac{5}{3} \quad [\text{Euc. vi. 3}];$$

$$\therefore \frac{x^2+40^2}{x^2} = \frac{25}{9}; \text{ whence } x=30;$$

thus the width of the road is 30 ft.

2. Let C be the position of the observer, B the top of the statue, A the foot of the column.

Then if $CB=x$ feet, we have

$$\frac{x}{CA} = \frac{a}{3a} = \frac{1}{3} \quad [\text{Euc. vi. 3}],$$

that is,
$$\frac{x}{\sqrt{x^2+16a^2}} = \frac{1}{3}; \text{ whence } x=a\sqrt{2}.$$

3. See figure on page 189.

Let BL be the flagstaff, LA the tower, C the observer;

then $BL=a$, $LA=b$, $DA=CE=d$, $EA=CD=h$;

also $BC^2=CE^2+EB^2=d^2+(a+b-h)^2$,

$$CA^2=CE^2+EA^2=d^2+h^2.$$

Now
$$\frac{BC}{CA} = \frac{BL}{LA}; \therefore \frac{(a+b-h)^2+d^2}{h^2+d^2} = \frac{a^2}{b^2};$$

$$\therefore \frac{(a+b)^2-2h(a+b)}{h^2+d^2} = \frac{a^2-b^2}{b^2};$$

or
$$\frac{a+b-2h}{h^2+d^2} = \frac{a-b}{b^2};$$

whence
$$(a-b)d^2=(a+b)b^2-2b^2h-(a-b)h^2.$$

4. See figure on page 190.

From O , the centre of the circle, draw OL , OM perpendicular to AB and DC respectively; then L , M bisect AB , DC . Let $DC=2x$ feet.

Then
$$\angle \beta = \frac{1}{2} \angle DOC \text{ at centre} = \angle COM.$$

Now
$$CD=2x=2CM=2OM \tan \beta$$

$$=2EL \tan \beta = (a+b) \tan \beta,$$

since
$$2EL=EB+EA=a+b.$$

5. With the same figure and notation as in the last Example, we have $ED=AB=20$ ft., and $\beta=45^\circ$.

$$\therefore 2x=(EB+EA) \tan 45^\circ=2EB+20;$$

$$\therefore x=EB+10.$$

Again $ED \cdot EC = EB \cdot EA = EB (EB + 20)$.

$$\therefore 20(20 + 2x) = (x - 10)(x + 10);$$

whence

$$x = 50.$$

Thus the height of the column is 100 ft.

6. Take the figure on page 190, interchanging the letters A and B . Then we have

$$\angle BDA = \angle BCA = \alpha - \beta; \quad \angle ABD = \angle ACE = 90^\circ - \alpha;$$

$$\therefore \angle DBC = \beta - \angle ABD = \alpha + \beta - 90^\circ.$$

Now from $\triangle CBD$,

$$CD = \frac{BD \sin DBC}{\sin BCE} = \frac{BD \sin (\alpha + \beta - 90^\circ)}{\sin BCE};$$

$$BD = \frac{AB \sin BAD}{\sin BDA} = \frac{a \sin BCE}{\sin (\alpha - \beta)};$$

$$\therefore CD = a \sin (\alpha + \beta - 90^\circ) \operatorname{cosec} (\alpha - \beta).$$

7. See figure on page 191.

Let CB be the pillar, BA the pedestal, E the point where the pillar subtends its maximum angle 30° .

Then using the same construction as in Ex. III. page 191, we have

$$\alpha = 30^\circ, \quad EA = 60 \text{ ft.}$$

$$\text{Now} \quad \angle AEB = \angle ECB = \frac{1}{2} \angle EDB = \frac{1}{2} (90^\circ - 30^\circ) = 30^\circ.$$

$$CB = 2CF = 2DF \tan 30^\circ = 2 \times 60 \times \frac{1}{\sqrt{3}} = 40 \sqrt{3} \text{ ft.}$$

8. Let O, P be the two positions of the observer; let

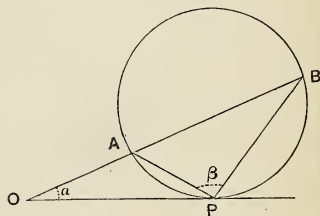
$$\angle APO = \theta.$$

Then $\angle ABP$, in alternate segment, $= \theta$.

$$\begin{aligned} \text{Now } AB &= \frac{AP \sin \beta}{\sin \theta} \\ &= \frac{c \sin \alpha \sin \beta}{\sin \theta \sin (\alpha + \theta)} \\ &= \frac{2c \sin \alpha \sin \beta}{\cos \alpha - \cos (\alpha + 2\theta)}. \end{aligned}$$

But, from $\triangle OPB$, $\alpha + 2\theta + \beta = 180^\circ$;

$$\therefore AB = 2c \sin \alpha \sin \beta / (\cos \alpha + \cos \beta).$$



9. Let OA be the tower, AB the flagstaff, P, Q the points at which the flagstaff subtends equal angles, R the point at which it subtends the greatest possible angle; then since $\angle APB = \angle AQB$;

$$\therefore B, A, P, Q \text{ are concyclic and } OP \cdot OQ = OA \cdot OB.$$

Again since ARB is the greatest angle subtended at a point in OQ by the str. line AB ; $\therefore$ a circle can be drawn to pass through A, B and touch OQ at R ;

$$\therefore OA \cdot OB = OR^2;$$

$$\therefore OP \cdot OQ = OR^2;$$

that is OP, OR, OQ are in geometrical progression.

10. Here $ABCD$ is a cyclic quadrilateral;

$$\therefore \frac{AB}{\sin ADB} = \frac{BD}{\sin BAD} = \frac{BD}{\sin BCD} = \frac{DC}{\sin CBD};$$

$$\therefore AB \sin CBD = CD \sin ADB.$$

11. Since $ABED$ is a cyclic quadrilateral, we have

$$\angle ADC = \angle EBC = \gamma, \text{ and } \angle BDC = \beta.$$

Also $\angle BDA = \gamma - \beta$, and $\angle ACE = \pi - (\alpha + \beta + \gamma)$.

$$\therefore BC = \frac{BD \sin \beta}{\sin (\alpha + \beta + \gamma)} = \frac{AB \sin (\alpha + \beta) \sin \beta}{\sin (\gamma - \beta) \sin (\alpha + \beta + \gamma)}.$$

12. Here the points P, Q, R, S, A are concyclic, and

$$\angle RAS = \angle PAQ,$$

since AR, AS are perpendicular to AP, AQ ;

$$\therefore PQ = RS = \sqrt{400 + 100 - 2 \times 200 \cos 30^\circ}$$

$$= \sqrt{500 - 200\sqrt{3}} = 12.4 \text{ ft. nearly.}$$

13. Let A, B be the two beacons, P, Q the positions of the ship at the end of 3 min. and 21 min. respectively. Let $\angle ABP = \alpha$, $\angle PBQ = \theta$. Then it is easily seen that

$$\angle OAP = 90^\circ + \alpha,$$

$$\angle OAQ = 90^\circ + \alpha + \theta.$$

Also from the $\triangle OBQ$ we have

$$\alpha + \theta + 90^\circ + \alpha = 135^\circ,$$

so that

$$\alpha + \theta = 45^\circ - \alpha.$$

If

$$PO = x, OQ = 7x;$$

hence

$$x = \frac{AP}{\sin 45^\circ} \sin (90^\circ + \alpha)$$

$$= AB \sqrt{2} \cdot \sin \alpha \cos \alpha$$

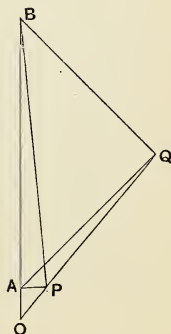
$$= \frac{5}{\sqrt{2}} \sin 2\alpha \dots\dots\dots(1).$$

Again

$$7x = \frac{AQ}{\sin 45^\circ} \sin (90^\circ + \alpha + \theta)$$

$$= AB \sqrt{2} \sin (\alpha + \theta) \cos (\alpha + \theta)$$

$$= \frac{5}{\sqrt{2}} \sin 2(\alpha + \theta) = \frac{5}{\sqrt{2}} \sin (90 - 2\alpha) = \frac{5}{\sqrt{2}} \cos 2\alpha \dots\dots\dots(2).$$



Squaring and adding (1) and (2), we have

$$50x^2 = \frac{25}{2}; \quad \therefore x = \frac{1}{2} \text{ mile.}$$

$\therefore$ the ship sails a mile in 6 min., or at the rate of 10 miles an hour.

Again if y miles be the distance from O at which the beacons subtend the greatest angle, we have

$$y^2 = OP \cdot OQ = \frac{7}{4}; \quad \therefore y = \frac{\sqrt{7}}{2}.$$

And the ship will travel this distance in $\frac{\sqrt{7}}{2} \times \frac{60}{10}$, or $3\sqrt{7}$ minutes.

14. Let AB be the flag-staff, BC the tower, D and E the first and second positions of the observer respectively.

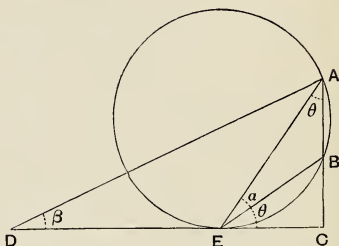
Then since AB subtends the maximum angle at E , the circle round ABE touches DC at E , so that

$\angle BEC = \angle EAB = \theta$, suppose.

$$\therefore \alpha + \theta = \angle EBC = 90^\circ - \theta;$$

$$\therefore 2\theta = 90^\circ - \alpha;$$

$$\begin{aligned} \text{also } \angle EAD &= \alpha + \theta - \beta \\ &= 90^\circ - \theta - \beta. \end{aligned}$$



$$\text{Now } AB = \frac{AE \sin \alpha}{\sin(\theta + \alpha)}, \quad AE = \frac{ED \sin \beta}{\sin(90^\circ - \theta - \beta)} = \frac{a \sin \beta}{\cos(\theta + \beta)};$$

$$\begin{aligned} \therefore AB &= \frac{a \sin \alpha \sin \beta}{\sin(\theta + \alpha) \cos(\theta + \beta)} = \frac{2a \sin \alpha \sin \beta}{\sin(2\theta + \alpha + \beta) + \sin(\alpha - \beta)} \\ &= \frac{2a \sin \alpha \sin \beta}{\cos \beta + \sin(\alpha - \beta)}. \end{aligned}$$

EXAMPLES. XVII. c. PAGE 195.

1. Let A be the top of the hill and B its projection on the horizontal plane through P, Q .

Let
then

$$AB = x \text{ yards};$$

$$BP = BA = x,$$

$$BQ = BA \cot 30^\circ = x\sqrt{3};$$

$$\therefore 3x^2 = x^2 + 500^2;$$

$$\therefore x = 250\sqrt{2};$$

that is, height of the hill = $250\sqrt{2}$ yards = 1060.5 feet.

Also

$$\begin{aligned}x : BN = QD : DN &= 6 : 30; \\ \therefore BD &= BN - DN = 5x - 30; \\ \therefore 25(x - 6)^2 &= 300^2 + 16(x - 6)^2; \\ \therefore 9(x - 6)^2 &= 300^2;\end{aligned}$$

whence

$$x = 106 \text{ or } -94;$$

$\therefore$ the light is 106 feet from the ground.

6. Let A be the balloon, and C, B, D the points of observation at which the angles of elevation of the balloon are $60^\circ, 30^\circ, 45^\circ$ respectively.

Let x yards be the height of the balloon;

then $AB = x \operatorname{cosec} 30^\circ = 2x$, $AC = x \operatorname{cosec} 60^\circ = \frac{2x}{\sqrt{3}}$, $AD = x \operatorname{cosec} 45^\circ = x\sqrt{2}$.

Now

$$2AD^2 + 2BD^2 = AB^2 + AC^2;$$

$$\therefore 4x^2 + 2 \times 880^2 = 4x^2 + \frac{4x^2}{3};$$

whence

$$x = 880 \cdot \sqrt{\frac{3}{2}} = 440\sqrt{6};$$

$\therefore$ height of balloon $= 440\sqrt{6}$ yards.

7. Let A be the top of the mountain, BC the base of length $2a$, D its middle point.

Let x be the height of the mountain;

then

$$AB = AC = x \operatorname{cosec} \theta,$$

$$AD = x \operatorname{cosec} \phi.$$

And we have

$$AD^2 + BD^2 = AB^2;$$

$$\therefore x^2 \operatorname{cosec}^2 \phi + a^2 = x^2 \operatorname{cosec}^2 \theta;$$

$$\therefore x^2 = \frac{a^2}{\operatorname{cosec}^2 \theta - \operatorname{cosec}^2 \phi} = \frac{a^2 \sin^2 \theta \cos^2 \theta}{\sin^2 \phi - \sin^2 \theta},$$

$$\therefore x = \frac{a \sin \theta \cos \theta}{\sqrt{\sin(\phi + \theta) \sin(\phi - \theta)}};$$

$\therefore$ height of mountain is $a \sin \theta \cos \theta \sqrt{\operatorname{cosec}(\phi + \theta) \operatorname{cosec}(\phi - \theta)}$.

8. Let AB, CD be the two vertical poles, E the point in the line BD joining their feet at which each subtends an angle α , and F any point in the horizontal plane such that $\angle DFB$ is a right angle;

then

$$\angle CED = \angle AEB = \alpha, \quad \angle AFB = \beta, \quad \angle CED = \gamma;$$

and

$$EB = AB \cot AEB = a \cot \alpha, \quad ED = b \cot \alpha;$$

$$\therefore BD = (a + b) \cot \alpha;$$

$$BF = AB \cot AFB = a \cot \beta, \quad DF = b \cot \gamma;$$

and

$$BD^2 = BF^2 + DF^2;$$

$$\therefore (a + b)^2 \cot^2 \alpha = a^2 \cot^2 \beta + b^2 \cot^2 \gamma.$$

9. Let A be the top of the hill, B its projection on the horizontal plane in which the road lies, C, D, E the three consecutive milestones whose angles of depression are observed.

Then $\angle ACB = \alpha$, $\angle ADB = \beta$, $\angle AEB = \gamma$,
and $ED = DC = 1760 \times 3$ feet.

Let x feet be the height of the hill;

then since $BC^2 + BE^2 = 2BD^2 + 2DE^2$;

$$\therefore x^2 \cot^2 \alpha + x^2 \cot^2 \gamma = 2x^2 \cot^2 \beta + 2 \times 1760^2 \times 3^2;$$

$$\therefore x = \frac{5280\sqrt{2}}{\sqrt{\cot^2 \alpha - 2 \cot^2 \beta + \cot^2 \gamma}}.$$

10. Let P, Q be the two positions of the observer.

Let x feet be the height of the chimneys;

then $AP = x \cot 60^\circ = \frac{x}{\sqrt{3}}$, $AQ = AB = x$;

$$\therefore x^2 = \frac{x^2}{3} + 80^2;$$

whence $x = 40\sqrt{6}$; $\therefore$ height of chimney $= 40\sqrt{6}$ feet.

Again $CQ = x \cot 30^\circ = x\sqrt{3}$;

$$\therefore CP = \sqrt{CQ^2 - PQ^2} = \sqrt{3 \times 40^2 \times 6 - 80^2} = 40\sqrt{14};$$

$$\therefore AC = 40\sqrt{14} + 40\sqrt{2} = 40(\sqrt{14} + \sqrt{2}) = 206 \text{ feet nearly.}$$

11. Let A be the balloon, C, D the positions of the two observers, B the point vertically below A in the horizontal plane of the observers.

Let $AB = x$ yds. = height of the balloon;

then $CD = 500$ yards,

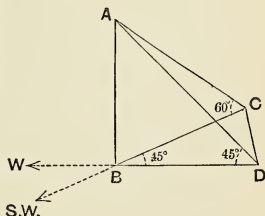
$$BC = x \cot 60^\circ = \frac{x}{\sqrt{3}}, \quad BD = x;$$

and $CD^2 = BC^2 + BD^2 - 2BC \cdot BD \cos 45^\circ$;

$$\therefore 500^2 = \frac{x^2}{3} + x^2 - \frac{x^2\sqrt{6}}{3};$$

whence $x^2 = 50^2(120 + 30\sqrt{6})$;

$$\therefore \text{height of balloon} = 50\sqrt{120 + 30\sqrt{6}} = 696 \text{ yds. nearly.}$$



12. In the diagram on page 195, let AC be a line of greatest slope on the hill, and let CH be the railway.

Then $\angle ACB = \alpha$, $\angle BCG = x$, $\angle HCG = \beta$, and $\angle CBG$ is a right angle. Let $AB = HG = h$.

Then $BC = h \cot \alpha$, $CG = h \cot \beta$.

$$\therefore \cos x = \frac{BC}{CG} = \cot \alpha \tan \beta.$$

EXAMPLES. XVII. d. PAGE 197.

1. Let C_1, C_2 be the positions of the two churches, M the position of the man. Then MC_1C_2 is an isosceles triangle, and the vertical angle at $M = 11^\circ 24'$.

If x miles is the height of the balloon, we have

$$\begin{aligned} x &= \frac{1}{2} \cot 5^\circ 42' \\ &= \frac{1}{2} \times 10.02 \\ &= 5.01. \end{aligned}$$

2. Let DC be the pole, then from a figure we have

$$DC = BC \sin 10^\circ 45' = \frac{100 \sin 5^\circ 30' \times \sin 10^\circ 45'}{\sin 5^\circ 15'}.$$

$$\log 100 + \log \sin 5^\circ 30' = .9816$$

$$\log \sin 10^\circ 45' = \bar{1}.2707$$

$$.2523$$

$$\log \sin 5^\circ 15' = \bar{2}.9612$$

$$\log DC = 1.2911$$

$$\therefore DC = 19.54 \text{ yards.}$$

Again,

$$BC = \frac{DC}{\tan 10^\circ 45'}.$$

$$\log DC = 1.2911$$

$$\log \tan 10^\circ 45' = \bar{1}.2785$$

$$\log BC = 2.0126$$

$$\therefore BC = 102.9 \text{ yards.}$$

3. If the required height is x feet, we have

$$\begin{aligned}
 x &= \frac{500 \sin 26^\circ 34' \times \sin 12^\circ 32'}{\sin 14^\circ 2'} \\
 \log 500 &= 2.6990 \\
 \log \sin 26^\circ 34' &= \bar{1}.6505 \\
 \log \sin 12^\circ 32' &= \bar{1}.3364 \\
 &\quad \underline{1.6859} \\
 \log \sin 14^\circ 2' &= \bar{1}.3847 \\
 \log x &= 2.3012 \\
 \therefore x &= 200.1.
 \end{aligned}$$

4. Let B be the position of the boat, FC the flagstaff, CD the cliff, then, if $CD = x$ feet, we have

$$\begin{aligned}
 x &= \frac{30 \sin 43^\circ 46'}{\sin 2^\circ 6'} \times \sin 44^\circ 8'. \\
 \log 30 &= 1.4771 \\
 \log \sin 43^\circ 46' &= \bar{1}.8399 \\
 \log \sin 44^\circ 8' &= \bar{1}.8429 \\
 &\quad \underline{1.1599} \\
 \log \sin 2^\circ 6' &= \bar{2}.5640 \\
 \log x &= 2.5959 \\
 \therefore \text{height} &= 394.4 \text{ ft.}
 \end{aligned}$$

Again,

$$\begin{aligned}
 BD &= \frac{x}{\tan 44^\circ 8'}. \\
 \log x &= 2.5959 \\
 \log \tan 44^\circ 8' &= \bar{1}.9869 \\
 \log BD &= 2.6090 \\
 \therefore \text{distance} &= 406.4 \text{ ft.}
 \end{aligned}$$

5. With the figure on page 184, we have

$$\angle PBC = 10^\circ, \angle PAC = 5^\circ; PB = AB = 5280 \text{ ft.}$$

Now

$$PC = PB \sin 10^\circ = 5280 \sin 10^\circ.$$

$$\begin{aligned}
 \log 5280 &= 3.7226 \\
 \log \sin 10^\circ &= \bar{1}.2397 \\
 &\quad \underline{2.9623}
 \end{aligned}$$

$$\therefore PC = 916.8 \text{ ft.}$$

Again,

$$\begin{aligned}
 BC &= BP \cos 10^\circ \\
 &= (1 \times \cos 10^\circ) \text{ miles} \\
 &= .9848 \text{ miles.}
 \end{aligned}$$

6. Let B be the point in the road which is vertically below the observer A . Let DC be the telegraph post, and let a horizontal line through A meet DC in E . Then $\angle DAE = 17^\circ 19'$, $\angle ACB = 8^\circ 36'$.

$$AE = BC = \frac{15}{\tan 8^\circ 36'}.$$

$$\log 15 = 1.1761$$

$$\log \tan 8^\circ 36' = \bar{1}.1797$$

$$\log AE = 1.9964$$

$$\therefore AE = 99.17 \text{ ft.}$$

Again,

$$DE = EA \tan 17^\circ 19'.$$

$$\log EA = 1.9964$$

$$\log \tan 17^\circ 19' = \bar{1}.4938$$

$$\log DE = 1.4902$$

$$\therefore DE = 30.91 \text{ ft.,}$$

$$\text{and } DC = 45.91 \text{ ft.}$$

7. Here we may take the third figure on page 131. Then $AC = 60$ miles, $CB_2 = CB_1 = 30$ miles, $\angle CAB_2 = 20^\circ 16'$.

$$\sin B = \frac{60}{30} \sin 20^\circ 16' = 2 \times .3464 = .6928;$$

$$\therefore B = 43^\circ 51', \text{ or } 136^\circ 9' \text{ (since } a < b \text{)}.$$

$$\therefore \angle ACB_2 = 23^\circ 35', \angle ACB_1 = 180^\circ - 64^\circ 7'.$$

Now

$$AB_2 = \frac{30 \sin 23^\circ 35'}{\sin 20^\circ 16'}.$$

$$\log 30 = 1.4771$$

$$\log \sin 23^\circ 35' = \bar{1}.6022$$

$$1.0793$$

$$\log \sin 20^\circ 16' = \bar{1}.5396$$

$$\log AB_2 = 1.5397$$

$$\therefore AB_2 = 34.65 \text{ miles.}$$

$$\begin{aligned}
 \text{Again,} \quad AB_1 &= \frac{30 \sin 64^\circ 7'}{\sin 20^\circ 16'} \\
 \log 30 &= 1.4771 \\
 \log \sin 64^\circ 7' &= \overline{1.9541} \\
 &\quad \underline{1.4312} \\
 \log \sin 20^\circ 16' &= \overline{1.5396} \\
 \log AB_1 &= 1.8916 \\
 \therefore AB_1 &= 77.91 \text{ miles.}
 \end{aligned}$$

Thus the train must travel at the rate of 11.55 miles or 25.97 miles per hour.

8. Let C be the doorstep, E the point of observation on the roof; then $CE = h$. Let AB be the spire and let ED drawn horizontally meet AB in D ; then $\angle ACB = 5a$, $\angle AED = 4a$. Also $\angle EAC = a$. Let $AB = x$.

$$\begin{aligned}
 \text{Then} \quad x = AC \sin 5a &= \frac{h \sin (90^\circ + 4a)}{\sin a} \cdot \sin 5a \\
 &= h \operatorname{cosec} a \cos 4a \sin 5a. \\
 CB &= x \cot 5a = h \operatorname{cosec} a \cos 4a \cos 5a.
 \end{aligned}$$

In the particular case

$$\alpha = 7^\circ 19', \quad 4a = 29^\circ 16', \quad 5a = 36^\circ 35'.$$

$$\begin{aligned}
 \text{Then} \quad x &= \frac{39 \cos 29^\circ 16' \times \sin 36^\circ 35'}{\sin 7^\circ 19'}. \\
 \log 39 &= 1.5911 \\
 \log \cos 29^\circ 16' &= \overline{1.9407} \\
 \log \sin 36^\circ 35' &= \overline{1.7753} \\
 &\quad \underline{1.3071} \\
 \log \sin 7^\circ 19' &= \overline{1.1050} \\
 \log x &= 2.2021 \\
 \therefore x &= 159.2; \text{ thus the height is } 159.2 \text{ ft.}
 \end{aligned}$$

$$\begin{aligned}
 \text{Again,} \quad CB &= \frac{x}{\tan 5a} = \frac{x}{\tan 36^\circ 35'}. \\
 \log x &= 2.2021 \\
 \log \tan 36^\circ 35' &= \overline{1.8705} \\
 \log CB &= 2.3316
 \end{aligned}$$

Thus the distance is 214.5 ft.

EXAMPLES. XVIII. a. PAGE 206.

1. $\text{Area} = \frac{1}{2} \times 300 \times 120 \sin 150^\circ = 300 \times 60 \times \frac{1}{2} = 9000 \text{ sq. feet.}$

2. $2s = 171 + 204 + 195 = 570;$

$$\therefore \text{area} = \sqrt{285 \times 114 \times 81 \times 90} = \sqrt{10^3 \times 57^2 \times 27^2} = 15390.$$

3. Let $a = 70$, $b = 147$, $c = 119$; then $s = 168$.

$$\therefore \sin B = \frac{2}{70 \times 119} \sqrt{168 \times 98 \times 21 \times 49} = \frac{2 \times 12 \times 7 \times 49}{70 \times 119} = \frac{84}{85}.$$

4. Let $a = 39$, $b = 40$, $c = 25$, and denote the perpendiculars by p_1 , p_2 , p_3 .

Then $\text{area} = \sqrt{52 \times 13 \times 12 \times 27} = 12 \times 13 \times 3.$

$$\therefore p_1 = \frac{2\Delta}{a} = 24, \quad p_2 = \frac{2\Delta}{b} = \frac{117}{5}, \quad p_3 = \frac{2\Delta}{c} = \frac{936}{25}.$$

5. $\text{Area} = \frac{30^2 \sin 22\frac{1}{2}^\circ \sin 112\frac{1}{2}^\circ}{2 \sin 135^\circ} = \frac{30^2 \sin 22\frac{1}{2}^\circ \cos 22\frac{1}{2}^\circ}{2 \sin 45^\circ} = \frac{30^2}{4} = 225 \text{ sq. ft.}$

6. The diagonal bisects the parallelogram;

$$\therefore \text{area} = 42 \times 32 \sin 30^\circ = 672 \text{ sq. feet.}$$

7. Let a yds. be the length of a side.

Then $\text{area} = a^2 \sin 150^\circ = \frac{a^2}{2};$

$$\therefore \frac{a^2}{2} = 648; \therefore a = 36 \text{ yards.}$$

$$\therefore \text{length of a side is } 36 \text{ yds.}$$

8. $13 + 14 + 15 = 42;$

$$\therefore \Delta = \sqrt{21 \times 8 \times 7 \times 6} = 4 \times 3 \times 7.$$

$$\therefore R = \frac{13 \times 14 \times 15}{4\Delta} = \frac{65}{8} = 8\frac{1}{8}.$$

$$r = \frac{\Delta}{21} = 4.$$

9. $17 + 10 + 21 = 48;$

$$\therefore \Delta = \sqrt{24 \times 7 \times 14 \times 3} = 4 \times 3 \times 7.$$

$$\therefore r_1 = \frac{\Delta}{7} = 12, \quad r_2 = \frac{\Delta}{14} = 6, \quad r_3 = \frac{\Delta}{3} = 28.$$

10. $s - a = \frac{\Delta}{r_1} = 12; \quad s - b = \frac{\Delta}{r_2} = 8; \quad s - c = \frac{\Delta}{r_3} = 4; \therefore s = 24;$

$$\therefore a = 12, \quad b = 16, \quad c = 20.$$

$$11. \sqrt{rr_1r_2r_3} = \sqrt{\frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-b} \cdot \frac{\Delta}{s-c}} = \Delta.$$

$$12. s(s-a) \tan \frac{A}{2} = s(s-a) \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{s(s-a)(s-b)(s-c)} = \Delta.$$

$$13. rr_1 \cot \frac{A}{2} = \frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \cdot \frac{s(s-a)}{\Delta} = \Delta.$$

$$14. 4Rrs = \frac{abc}{\Delta} \cdot \frac{\Delta}{s} \cdot s = abc.$$

$$15. r_1r_2r_3 = \frac{\Delta^3}{(s-a)(s-b)(s-c)} = \frac{\Delta^3s}{\Delta^2} = \Delta s = rs^2.$$

$$16. r \cot \frac{B}{2} \cot \frac{C}{2} = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}} \cdot \cot \frac{B}{2} \cot \frac{C}{2} = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}} = r_1.$$

$$17. \text{First side} = r \left(\frac{a}{2} + \frac{b}{2} + \frac{c}{2} \right) = rs = \Delta.$$

$$18. r_1r_2 + rr_3 = \frac{\Delta^2}{(s-a)(s-b)} + \frac{\Delta^2}{s(s-c)} = \frac{\Delta^2 \{s(s-c) + (s-a)(s-b)\}}{\Delta^2} \\ = 2s^2 - (a+b+c)s + ab = ab.$$

$$20. r_1 + r_2 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} \\ = 4R \cos \frac{C}{2} \cdot \sin \frac{A+B}{2} = 4R \cos^2 \frac{C}{2} \\ = \frac{2c}{\sin C} \cdot \cos^2 \frac{C}{2} = c \cot \frac{C}{2}.$$

$$21. \text{As in Ex. 20, } r_1 - r = 4R \sin \frac{A}{2} \cdot \cos \frac{B+C}{2} = 4R \sin^2 \frac{A}{2},$$

$$r_2 + r_3 = 4R \cos^2 \frac{A}{2}.$$

$$\therefore (r_1 - r)(r_2 + r_3) = 4R^2 \sin^2 A = a^2.$$

22. By the formulæ of Art. 212, we have

$$\frac{r_1}{r} = \cot \frac{B}{2} \cot \frac{C}{2}; \therefore r_1 \cot \frac{A}{2} = r \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

$$23. \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s-a+s-b+s-c}{\Delta} = \frac{3s-2s}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}.$$

$$24. \quad r_2 r_3 + r_3 r_1 + r_1 r_2 = \frac{\Delta^2}{(s-b)(s-c)} + \dots + \dots = s(s-a) + \dots + \dots \\ = 3s^2 - (a+b+c)s = s^2.$$

$$25. \quad \text{As in Ex. 21, } r_2 + r_3 = 4R \cos^2 \frac{A}{2}, \quad r_1 - r = 4R \sin^2 \frac{A}{2}. \\ \therefore r_1 + r_2 + r_3 - r = 4R \left(\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2} \right) = 4R.$$

$$26. \quad r_1 + r_2 = 4R \cos^2 \frac{C}{2}, \quad r_3 - r = 4R \sin^2 \frac{C}{2}. \quad [\text{Ex. 21.}]$$

$$\therefore r + r_1 + r_2 - r_3 = 4R \left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2} \right) = 4R \cos C.$$

$$27. \quad b^2 \sin 2C + c^2 \sin 2B = 2b \sin C \cdot b \cos C + 2c \sin B \cdot c \cos B \\ = 2b \sin C (b \cos C + c \cos B) = 2ab \sin C = 4\Delta.$$

$$28. \quad (a+b) \sec \frac{A-B}{2} = 2R (\sin A + \sin B) \sec \frac{A-B}{2} \\ = 4R \sin \frac{A+B}{2} \cos \frac{A-B}{2} \sec \frac{A-B}{2} = 4R \cos \frac{C}{2}.$$

$$29. \quad a^2 - b^2 = 4R^2 (\sin^2 A - \sin^2 B) = 4R^2 \sin (A+B) \sin (A-B) \\ = 4R^2 \sin C \sin (A-B) = 2Rc \sin (A-B).$$

$$30. \quad \text{First side} = 4R^2 \cdot \frac{\sin^2 A - \sin^2 B}{2} \cdot \frac{\sin A \sin B}{\sin (A-B)} \\ = \frac{4R^2 \sin (A+B) \sin A \sin B}{2} \\ = \frac{2R \sin A \cdot 2R \sin B \cdot \sin C}{2} = \frac{ab}{2} \sin C = \Delta.$$

$$31. \quad (1) \text{ We have } 2\Delta = ap_1 = bp_2 = cp_3;$$

$$\therefore \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{a+b+c}{2\Delta} = \frac{s}{\Delta} = \frac{1}{r}.$$

$$(2) \quad \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{a+b-c}{2\Delta} = \frac{s-c}{\Delta} = \frac{1}{r_3}.$$

$$32. \quad (r_1 - r)(r_2 - r)(r_3 - r) = 64R^3 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} \quad [\text{Ex. 21}] \\ = 4Rr^2. \quad [\text{Art. 212.}]$$

$$\begin{aligned}
 33. \quad \left(\frac{1}{r} - \frac{1}{r_1}\right) \left(\frac{1}{r} - \frac{1}{r_2}\right) \left(\frac{1}{r} - \frac{1}{r_3}\right) &= \frac{(r_1 - r)(r_2 - r)(r_3 - r)}{r^3 r_1 r_2 r_3} = \frac{4Rr^2}{r^2 \Delta^2} \quad [\text{Ex. 11}] \\
 &= \frac{4R}{r^2 s^2}.
 \end{aligned}$$

$$\begin{aligned}
 34. \quad 4\Delta (\cot A + \cot B + \cot C) &= 2bc \cos A + 2ca \cos B + 2ab \cos C \\
 &= b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2 \\
 &= a^2 + b^2 + c^2.
 \end{aligned}$$

$$\begin{aligned}
 35. \quad \frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} &= \frac{(b-c)(s-a) + (c-a)(s-b) + (a-b)(s-c)}{\Delta} \\
 &= \frac{s(b-c+c-a+a-b) - \{a(b-c) + b(c-a) + c(a-b)\}}{\Delta} = 0.
 \end{aligned}$$

$$\begin{aligned}
 36. \quad a^2 b^2 c^2 (\sin 2A + \sin 2B + \sin 2C) &= 4a^2 b^2 c^2 \sin A \sin B \sin C \\
 &= 4 \cdot bc \sin A \cdot ca \sin B \cdot ab \sin C \\
 &= 32\Delta^3.
 \end{aligned}$$

$$\begin{aligned}
 37. \quad a \cos A + b \cos B + c \cos C &= 2R (\sin A \cos A + \sin B \cos B + \sin C \cos C) \\
 &= R (\sin 2A + \sin 2B + \sin 2C) \\
 &= 4R \sin A \sin B \sin C.
 \end{aligned}$$

$$\begin{aligned}
 38. \quad a \cot A + b \cot B + c \cot C &= 2R (\cos A + \cos B + \cos C) \\
 &= 2R \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}\right) \\
 &= 2(R+r).
 \end{aligned}$$

$$\begin{aligned}
 39. \quad (b+c) \tan \frac{A}{2} &= 2R (\sin B + \sin C) \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = 4R \cos \frac{B-C}{2} \cos \frac{B+C}{2} \\
 &= 2R (\cos B + \cos C);
 \end{aligned}$$

$$\therefore (b+c) \tan \frac{A}{2} + \text{two similar terms} = 4R (\cos A + \cos B + \cos C).$$

$$\begin{aligned}
 40. \quad r (\sin A + \sin B + \sin C) &= 4r \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \\
 &= 16R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \\
 &= 2R \sin A \sin B \sin C.
 \end{aligned}$$

$$\begin{aligned}
 41. \quad \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} &= \frac{1}{2} (3 + \cos A + \cos B + \cos C) \\
 &= \frac{1}{2} \left(4 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) \\
 &= 2 + \frac{r}{2R}.
 \end{aligned}$$

EXAMPLES. XVIII. b. PAGE 210.

$$\begin{aligned}
 1. \quad \text{By Art. 214, the area} &= \frac{1}{2} \cdot 10 \cdot 9 \cdot \sin 36^\circ \\
 &= 45 \times \cdot 588 = 26 \cdot 46 \text{ sq. ft.}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \text{By Art. 215, the perimeter} &= 2 \cdot 15 \cdot \frac{3}{2} \cdot \tan 12^\circ \\
 &= 45 \times \cdot 213 = 9 \cdot 585 \text{ yds.}
 \end{aligned}$$

$$\text{The area} = 15 \cdot \frac{9}{4} \cdot \tan 12^\circ = \frac{135 \times \cdot 213}{4} = 7 \cdot 18875 \text{ sq. yds.}$$

$$\begin{aligned}
 3. \quad \text{In fig. of Art. 215 let } AB &= 2x = \text{side of regular hexagon. Then} \\
 OD &= x \cot 30^\circ = x\sqrt{3}. \\
 \therefore \text{area of inscribed circle} &= 3\pi x^2.
 \end{aligned}$$

$$\begin{aligned}
 \text{Again, in fig. of Art. 214, if } AB &\text{ be the side of the hexagon,} \\
 OA &= AD \operatorname{cosec} 30^\circ = 2x. \\
 \therefore \text{area of circumscribed circle} &= 4\pi x^2. \\
 \therefore \text{ratio of areas is } 3 &\text{ to } 4.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \text{With fig. and notation of Art. 217,} \\
 \text{area of pentagon} &= 5AD \cdot OD = 5r^2 \tan 36^\circ; \\
 \therefore 250 &= 5r^2 \tan 36^\circ; \pi r^2 = \frac{22}{7} \times 50 \cot 36^\circ = 216 \cdot 23 \text{ sq. ft.}
 \end{aligned}$$

$$5. \quad \text{Area of circle} = \pi r^2 = 1386; \text{ whence } r = 21.$$

$$\text{By Art. 214, perimeter} = 16r \sin 22^\circ 30' = 16 \times 21 \times \cdot 382 = 128 \cdot 352 \text{ in.}$$

$$6. \quad \text{Area of circle} = \pi r^2 = 616; \text{ whence } r = 14.$$

$$\begin{aligned}
 \text{By Art. 215, perimeter of pentagon} &= 2 \cdot 5 \cdot 14 \tan 36^\circ = 140 \times \cdot 727 \\
 &= 101 \cdot 78 \text{ ft.}
 \end{aligned}$$

$$7. \quad \text{Area of circle} = \pi r^2 = 2464; \text{ whence } r = 28.$$

Now in Art. 214, if AB is a side of the quindecagon, $OD = 28$, and diameter of required circle $= 2AO = 2OD \sec 12^\circ = 2 \times 28 \times 1 \cdot 022 = 57 \cdot 232 \text{ ft.}$

8. Let r be the radius of the circle; then

$$\text{for the pentagon,} \quad 50 = \frac{5}{2} r^2 \sin \frac{2\pi}{5},$$

$$\text{for the dodecagon,} \quad \text{area} = \frac{12}{2} r^2 \sin \frac{\pi}{6}; \quad [\text{Art. 214}]$$

$$\therefore \frac{\text{area of dodecagon}}{50} = \frac{12}{5} \cdot \frac{\sin 30^\circ}{\sin 72^\circ};$$

$$\therefore \text{area of dodecagon} = 60 \operatorname{cosec} 72^\circ = 60 \times 1.0515 = 63.09 \text{ sq. ft.}$$

9. Let the perimeters of pentagon and decagon be denoted by $10a$ and $10b$ respectively. Then as in Example 2, page 209,

$$\text{area of pentagon} = 5a^2 \cot \frac{\pi}{5},$$

$$\text{area of decagon} = \frac{10b^2}{4} \cot \frac{\pi}{10};$$

$$\therefore 2a^2 \cot 36^\circ = b^2 \cot 18^\circ.$$

$$\text{Now} \quad \cot^2 18^\circ = \frac{1 + \cos 36^\circ}{1 - \cos 36^\circ} = \frac{1 + \frac{\sqrt{5}+1}{4}}{1 - \frac{\sqrt{5}+1}{4}} = \frac{5 + \sqrt{5}}{3 - \sqrt{5}};$$

$$\text{and} \quad \cot^2 36^\circ = \frac{1 + \frac{\sqrt{5}-1}{4}}{1 - \frac{\sqrt{5}-1}{4}} = \frac{3 + \sqrt{5}}{5 - \sqrt{5}};$$

$$\therefore \frac{\cot^2 18^\circ}{\cot^2 36^\circ} = \frac{20}{4} = 5.$$

$$\therefore \frac{a^2}{b^2} = \frac{\sqrt{5}}{2}, \text{ or } \frac{a}{b} = \frac{\sqrt[4]{5}}{\sqrt{2}}.$$

10. Let $2na$ be the common perimeter, so that $2a$, a are sides respectively of the two polygons.

$$\text{Area of polygon of } n \text{ sides} = \frac{n}{4} \cdot 4a^2 \cot \frac{\pi}{n}.$$

$$\text{Area of polygon of } 2n \text{ sides} = \frac{2n}{4} \cdot a^2 \cot \frac{\pi}{2n};$$

$$\begin{aligned} \therefore \text{ratio of areas} &= \left(2 \cos \frac{\pi}{n} \sin \frac{\pi}{2n} \right) : \left(\sin \frac{\pi}{n} \cos \frac{\pi}{2n} \right) \\ &= \left(2 \cos \frac{\pi}{n} \sin \frac{\pi}{2n} \right) : \left(2 \sin \frac{\pi}{2n} \cos^2 \frac{\pi}{2n} \right) \\ &= \left(2 \cos \frac{\pi}{n} \right) : \left(1 + \cos \frac{\pi}{n} \right). \end{aligned}$$

11. In the fig. of Art. 214, if $AD=a$, $OA=R$, we have $R=a \operatorname{cosec} \frac{\pi}{n}$.

In the fig. of Art. 215, if $OD=r$, we have $r=a \cot \frac{\pi}{n}$.

$$\begin{aligned} \therefore R+r &= a \left(\frac{1}{\sin \frac{\pi}{n}} + \frac{\cos \frac{\pi}{n}}{\sin \frac{\pi}{n}} \right) \\ &= \frac{a \left(1 + \cos \frac{\pi}{n} \right)}{2 \sin \frac{\pi}{2n} \cdot \cos \frac{\pi}{2n}} = \frac{2a \cos^2 \frac{\pi}{2n}}{2 \sin \frac{\pi}{2n} \cdot \cos \frac{\pi}{2n}} = a \cot \frac{\pi}{2n}. \end{aligned}$$

12. Let p, h, d represent a side of the pentagon, hexagon, and decagon respectively inscribed in a circle of radius r ; then

$$p=2r \sin 36^\circ, \quad h=2r \sin 30^\circ, \quad d=2r \sin 18^\circ.$$

$$\begin{aligned} \therefore h^2+d^2 &= 4r^2 \left\{ \frac{1}{4} + \left(\frac{\sqrt{5}-1}{4} \right)^2 \right\} = 4r^2 \left(\frac{4+6-2\sqrt{5}}{16} \right) \\ &= 4r^2 \left(\frac{10-2\sqrt{5}}{16} \right) = 4r^2 \sin^2 36^\circ = p^2. \quad [\text{See Ex. Art. 126.}] \end{aligned}$$

$$13. \quad A_1 = \frac{1}{2} nr^2 \sin \frac{2\pi}{n}, \quad B_1 = nr^2 \tan \frac{\pi}{n}, \quad [\text{Arts. 214, 215}]$$

$$A_2 = \frac{1}{2} 2n \cdot r^2 \sin \frac{\pi}{n}, \quad B_2 = 2nr^2 \tan \frac{\pi}{2n}.$$

$$\therefore A_1 B_1 = \frac{1}{2} n^2 r^4 \cdot 2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} \tan \frac{\pi}{n} = n^2 r^4 \sin^2 \frac{\pi}{n} = A_2^2.$$

Thus A_2 is the geom. mean between A_1 and B_1 .

$$\begin{aligned} \text{Again} \quad \frac{1}{A_2} + \frac{1}{B_1} &= \frac{1}{nr^2 \sin \frac{\pi}{n}} + \frac{1}{nr^2 \tan \frac{\pi}{n}} = \frac{1 + \cos \frac{\pi}{n}}{nr^2 \sin \frac{\pi}{n}} \\ &= \frac{2 \cos^2 \frac{\pi}{2n}}{2nr^2 \sin \frac{\pi}{2n} \cos \frac{\pi}{2n}} = \frac{1}{nr^2 \tan \frac{\pi}{2n}} = \frac{2}{B_2}. \end{aligned}$$

Thus B_2 is the harm. mean between A_2 and B_1 .

EXAMPLES. XVIII. c. PAGE 218.

1. Required distance

$$= r \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{B}{2} \sin \frac{C}{2}.$$

2. $I_1 A = r_1 \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \operatorname{cosec} \frac{A}{2} = 4R \cos \frac{B}{2} \cos \frac{C}{2},$

$$I_1 B = r_1 \operatorname{cosec} \left(90^\circ - \frac{B}{2} \right) = r_1 \sec \frac{B}{2} = 4R \sin \frac{A}{2} \cos \frac{C}{2},$$

$$I_1 C = r_1 \operatorname{cosec} \left(90^\circ - \frac{C}{2} \right) = r_1 \sec \frac{C}{2} = 4R \sin \frac{A}{2} \cos \frac{B}{2}.$$

3. (1) From the fig. of Art. 219, we have

$$\begin{aligned} \text{area} &= \frac{1}{2} I_1 I_2 \cdot I_3 C = \frac{1}{2} 4R \cos \frac{C}{2} \cdot r_3 \operatorname{cosec} \frac{C}{2} \\ &= 2R r_3 \cot \frac{C}{2} = 2Rs. \end{aligned}$$

$$(2) \text{ Area} = 2Rs = 2R \cdot \frac{\Delta}{r} = \frac{1}{2} \Delta \operatorname{cosec} \frac{A}{2} \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2}. \quad [\text{Art. 212.}]$$

4. We have $II_1 = IC \operatorname{cosec} II_1 C = IC \operatorname{cosec} \frac{B}{2}.$

$$\begin{aligned} \therefore r II_1 \cdot II_2 \cdot II_3 &= 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \cdot IA \cdot IB \cdot IC \cdot \operatorname{cosec} \frac{A}{2} \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2} \\ &= 4R \cdot IA \cdot IB \cdot IC. \end{aligned}$$

5. Perimeter of pedal triangle $= R (\sin 2A + \sin 2B + \sin 2C)$
 $= 4R \sin A \sin B \sin C.$

In-radius of pedal triangle

$$\begin{aligned} &= 4 \cdot \frac{R}{2} \sin (90^\circ - A) \sin (90^\circ - B) \sin (90^\circ - C) \quad [\text{Arts. 225, 212}] \\ &= 2R \cos A \cos B \cos C. \end{aligned}$$

$$\begin{aligned} 6. (1) \quad \frac{g}{a^2} + \frac{h}{b^2} + \frac{k}{c^2} &= \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} \\ &= \frac{2bc \cos A + 2ca \cos B + 2ab \cos C}{2abc} \\ &= \frac{b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2}{2abc} = \frac{a^2 + b^2 + c^2}{2abc}. \end{aligned}$$

$$\begin{aligned}
 (2) \quad \frac{b^2 - c^2}{a^2} g + \dots + \dots &= \frac{b^2 - c^2}{a} \cos A + \dots + \dots \\
 &= \frac{(b^2 - c^2)(b^2 + c^2 - a^2)}{2abc} + \dots + \dots \\
 &= 0.
 \end{aligned}$$

7. Let ρ_1, ρ_2, ρ_3 be the radii, then

$$\rho_1 = 4 \frac{R}{2} \cdot \sin \frac{G}{2} \cos \frac{H}{2} \cos \frac{K}{2} = 2R \cos A \sin B \sin C. \quad [\text{Arts. 225, 212.}]$$

8. (1) $a \cos A, b \cos B, c \cos C, 180^\circ - 2A, 180^\circ - 2B, 180^\circ - 2C$ are the sides and angles of the pedal triangle of the triangle ABC . Hence in any formula connecting a, b, c, A, B, C , we may replace the sides by $a \cos A, b \cos B, c \cos C$ respectively, and the angles by $180^\circ - 2A, 180^\circ - 2B, 180^\circ - 2C$ respectively.

(2) $a \operatorname{cosec} \frac{A}{2}, b \operatorname{cosec} \frac{B}{2}, c \operatorname{cosec} \frac{C}{2}, 90^\circ - \frac{A}{2}, 90^\circ - \frac{B}{2}, 90^\circ - \frac{C}{2}$ are the sides and angles of the ex-central triangle of the triangle ABC . Hence the proposition follows.

9. We have $SI^2 = R^2 - 2Rr$ [Art. 228].

$\therefore R^2 - 2Rr$ must be positive; that is $R - 2r$ must be positive.

Hence R can never exceed $2r$.

10. If

$$R = 2r;$$

we have

$$SI^2 = R^2 - 2Rr = 0;$$

$\therefore$ the centres of the circumcircle and in-circle coincide, and hence the triangle is equilateral.

$$\begin{aligned}
 11. \quad SI^2 + SI_1^2 + SI_2^2 + SI_3^2 &= R^2 - 2Rr + R^2 + 2Rr_1 + R^2 + 2Rr_2 + R^2 + 2Rr_3 \\
 &= 4R^2 + 2R(r_1 + r_2 + r_3 - r) \\
 &= 12R^2 \quad [\text{XVIII. a. Ex. 25}].
 \end{aligned}$$

$$12. (1) \quad a \cdot AI^2 + b \cdot BI^2 + c \cdot CI^2 = r^2 a \operatorname{cosec}^2 \frac{A}{2} + \dots + \dots$$

$$\begin{aligned}
 &= abc r^2 \left\{ \frac{1}{(s-b)(s-c)} + \dots + \dots \right\} \\
 &= \frac{abc r^2 s}{(s-a)(s-b)(s-c)} = \frac{abc r^2 s^2}{\Delta^2} = abc.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad a \cdot AI_1^2 + b \cdot BI_1^2 + c \cdot CI_1^2 &= r_1^2 \left(a \operatorname{cosec}^2 \frac{A}{2} + b \sec^2 \frac{B}{2} + c \sec^2 \frac{C}{2} \right) \\
 &= abc r_1^2 \left\{ \frac{1}{(s-b)(s-c)} - \frac{1}{s(s-b)} - \frac{1}{s(s-c)} \right\} \\
 &= \frac{abc r_1^2 \{s - (s-c) - (s-b)\}}{s(s-b)(s-c)} = \frac{abc r_1^2 (s-a)^2}{\Delta^2} = abc.
 \end{aligned}$$

$$13. (1) \text{ We have } \frac{OG}{AG} = \frac{\Delta OBC}{\Delta ABC}.$$

$$\therefore \frac{OG}{AG} + \frac{OH}{BH} + \frac{OK}{CK} = \frac{\text{sum of areas of } \Delta^s OBC, OAC, OAB}{\Delta ABC} = 1.$$

(2) Since A, H, O, K are concyclic, $HK = AO \sin A$;

$$\text{thus } AO = \frac{a \cos A}{\sin A} = a \cot A.$$

$\therefore OG + a \cot A = OG + AO = AG$, and the result required is reduced to that already proved in (1).

$$14. \text{ Circum-radius of } \Delta AHK = \frac{KH}{2 \sin A} = \frac{R \sin 2A}{2 \sin A} = R \cos A;$$

$\therefore$ sum of circum-radii of $\Delta^s AHK, BKG, CGH = R(\cos A + \cos B + \cos C)$

$$= R \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)$$

$$= R + r.$$

$$15. \text{ We have } A_1 = \frac{\pi}{2} - \frac{A}{2},$$

$$A_2 = \frac{\pi}{2} - \frac{A_1}{2} = \frac{\pi}{2} - \frac{1}{2} \left(\frac{\pi}{2} - \frac{A}{2} \right),$$

$$A_3 = \frac{\pi}{2} - \frac{A_2}{2} = \frac{\pi}{2} - \frac{1}{2} \left\{ \frac{\pi}{2} - \frac{1}{2} \left(\frac{\pi}{2} - \frac{A}{2} \right) \right\};$$

.....

$$\therefore A_n = \frac{\pi}{2} \left(1 - \frac{1}{2} + \left(\frac{1}{2} \right)^2 - \dots \text{ to } n \text{ terms} \right) + (-1)^n \frac{A}{2^n}$$

$$= \frac{\pi}{2} \frac{1 - \left(-\frac{1}{2} \right)^n}{1 + \frac{1}{2}} + (-1)^n \frac{A}{2^n}$$

$$= \frac{\pi}{3} \left(1 - (-1)^n \frac{1}{2^n} \right) + (-1)^n \frac{A}{2^n}$$

$$= \frac{\pi}{3} + (-1)^n \frac{1}{2^n} \left(A - \frac{\pi}{3} \right);$$

similarly

$$B_n = \frac{\pi}{3} + (-1)^n \frac{1}{2^n} \left(B - \frac{\pi}{3} \right),$$

$$C_n = \frac{\pi}{3} + (-1)^n \frac{1}{2^n} \left(C - \frac{\pi}{3} \right).$$

When n is indefinitely increased, then $A_n = B_n = C_n = \frac{\pi}{3}$.

$$16. \quad (1) \quad OS^2 = R^2 - 8R^2 \cos A \cos B \cos C \quad [\text{Art. 230}]$$

$$\begin{aligned} &= R^2 + 2R^2 (1 + \cos 2A + \cos 2B + \cos 2C) \\ &= 9R^2 - 2R^2 (1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C) \\ &= 9R^2 - 4R^2 (\sin^2 A + \sin^2 B + \sin^2 C) \\ &= 9R^2 - a^2 - b^2 - c^2. \end{aligned}$$

$$(2) \quad \text{We have } AO = 2R \cos A; \quad IA = r \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{B}{2} \sin \frac{C}{2};$$

$$\angle IAO = (90^\circ - B) - \frac{A}{2} = \frac{C - B}{2};$$

$$\begin{aligned} \therefore OI^2 &= 4R^2 \cos^2 A + 16R^2 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} - 16R^2 \cos A \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{C - B}{2} \\ &= 4R^2 \left(\cos^2 A + 4 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} - \cos A \sin B \sin C - 4 \cos A \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} \right) \\ &= 4R^2 \left(\cos^2 A + 8 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} - \cos A \sin B \sin C \right) \\ &= 2r^2 - 4R^2 \cos A (\sin B \sin C - \cos A) \\ &= 2r^2 - 4R^2 \cos A \cos B \cos C. \end{aligned}$$

(3) We have

$$AO = 2R \cos A; \quad I_1A = r_1 \operatorname{cosec} \frac{A}{2} = 4R \cos \frac{B}{2} \cos \frac{C}{2}; \quad \angle I_1AO = \frac{C - B}{2};$$

$$\begin{aligned} \therefore OI_1^2 &= 4R^2 \cos^2 A + 16R^2 \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - 16R^2 \cos A \cos \frac{B}{2} \cos \frac{C}{2} \cos \frac{C - B}{2} \\ &= 4R^2 \left(\cos^2 A + 4 \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - 4 \cos A \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - \cos A \sin B \sin C \right) \\ &= 4R^2 \left(\cos^2 A + 8 \sin^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} - \cos A \sin B \sin C \right) \\ &= 2r_1^2 - 4R^2 \cos A (\sin B \sin C - \cos A) \\ &= 2r_1^2 - 4R^2 \cos A \cos B \cos C. \end{aligned}$$

17. Let R' , ρ , ρ_1 , ρ_2 , ρ_3 represent the circum-radius, the in-radius, and the ex-radii of the pedal triangle; then, since A , B , C are the ex-centres of the pedal triangle, we have, as in Art. 229,

$$f^2 = R'^2 + 2R'\rho_1; \text{ also } R' = \frac{R}{2}.$$

$$\begin{aligned} \therefore f^2 + g^2 + h^2 &= 3R'^2 + 2R'(\rho_1 + \rho_2 + \rho_3) \\ &= 3R'^2 + 2R'(4R' + \rho) \quad [\text{XVIII. a. Ex. 25}] \\ &= 11R'^2 + 2R'\rho \end{aligned}$$

$$\begin{aligned} &= 11R'^2 + 8R'^2 \sin \frac{G}{2} \sin \frac{H}{2} \sin \frac{F}{2} \\ &= 11R'^2 + 8R'^2 \cos A \cos B \cos C; \quad [\text{Art. 224}] \end{aligned}$$

$$\therefore 4(f^2 + g^2 + h^2) = 11R^2 + 8R^2 \cos A \cos B \cos C. \quad [\text{Art. 225.}]$$

EXAMPLES. XVIII. d. PAGE 223.

1. Let r be the radius, and a , b , c , d the sides of the quadrilateral.

Then we have $2S = ra + rb + rc + rd$;

$$\therefore r = \frac{S}{\sigma}.$$

2. Let $ABCD$ be the quadrilateral having sides

$$AB = BC = 3; \quad CD = DA = 4.$$

Then $\angle BAC = \angle BCA$; $\angle DAC = \angle DCA$;

$$\angle BAD + \angle BCD = 180^\circ; \quad \therefore \angle BAD = \angle BCD = 90^\circ,$$

and the $\triangle BAD$, BCD are identically equal.

Thus it easily follows that the bisectors of the $\angle BAD$, BCD meet on BD which bisects the angles ABC , ADC .

$\therefore$ a circle can be inscribed in the quadrilateral.

If r be its radius we have

$$r(3 + 3 + 4 + 4) = 2 \text{ (area of quadrilateral)} = 2 \cdot 4 \cdot 3.$$

$$\therefore r = \frac{12}{7} = 1\frac{5}{7}.$$

Also radius of circumscribed circle $= \frac{1}{2}BD = \frac{1}{2}\sqrt{3^2 + 4^2} = \frac{5}{2} = 2\frac{1}{2}.$

3. Let $ABCD$ be the quadrilateral, and let

$$AB=1, BC=2, CD=4, DA=3.$$

Then $AC^2=5-4\cos ABC=5+4\cos ADC$;

also $AC^2=25-24\cos ADC$;

$$\therefore 5+4\cos ADC=25-24\cos ADC.$$

$$\therefore \cos ADC=\frac{20}{28}=\frac{5}{7};$$

and area of quadrilateral $=\sqrt{4 \times 3 \times 1 \times 2}=\sqrt{24}$;

$$\therefore \text{radius of inscribed circle}=\frac{\sqrt{24}}{5}=.98 \text{ nearly.}$$

4. Let $ABCD$ be the quadrilateral, and let

$$AB=60, BC=25, CD=52, DA=39.$$

Then $AC^2=60^2+25^2-2 \cdot 60 \cdot 25 \cos ABC=52^2+39^2-2 \times 52 \times 39 \cos ADC=52^2+39^2+2 \times 52 \times 39 \cos ABC$;

also $AC^2=52^2+39^2-2 \times 52 \times 39 \cos ADC=52^2+39^2+2 \times 52 \times 39 \cos ABC$;

$$\therefore \cos ABC=0, \text{ that is } \angle ABC=90^\circ;$$

and hence the $\angle ADC=90^\circ$;

$$\therefore AC^2=60^2+25^2=52^2+39^2; \therefore AC=65;$$

$$\therefore BD=\frac{60 \times 52+39 \times 25}{65}=48+15=63.$$

Also the area $=\sqrt{28 \times 63 \times 36 \times 49}=62 \times 7^2=1764.$

5. Let $ABCD$ be the quadrilateral, and let

$$AB=4, BC=5, CD=8, DA=9.$$

Then since $AB+BC=9$, AC must be less than 9;

$$\therefore \text{diagonal } BD=9;$$

$$\text{and } \cos A=\frac{2}{9}; \therefore \sin A=\frac{\sqrt{77}}{9};$$

$$\cos C=\frac{1}{10}; \therefore \sin C=\frac{3\sqrt{11}}{10};$$

$$\therefore \text{area}=\frac{1}{2}(ad \sin A + bc \sin C)=18\frac{\sqrt{77}}{9}+20 \cdot \frac{3\sqrt{11}}{10}=2\sqrt{77}+6\sqrt{11}.$$

6. Since the quadrilateral is such that one circle can be inscribed in it and another circle circumscribed about it, therefore

$$\cos A = \frac{ad - bc}{ad + bc}, \text{ and } \sin A = \frac{2\sqrt{abcd}}{ad + bc}; \quad [\text{Art. 234, Ex.}]$$

$$\therefore \text{area} = \frac{1}{2} (ad \sin A + bc \sin C) = \sqrt{abcd}.$$

If r be radius of inscribed circle,

$$\frac{r}{2} (a + b + c + d) = \text{area} = \sqrt{abcd}.$$

$$\therefore r = \frac{2\sqrt{abcd}}{a + b + c + d}.$$

7. We have $S^2 = (\sigma - a)(\sigma - b)(\sigma - c)(\sigma - d) - abcd \cos^2 a$; [Art. 232]

$\therefore S$ is greatest when $\cos a = 0$, since σ, a, b, c, d are constant.

But a is half the sum of two opposite angles;

$\therefore$ area is a maximum when the sum of two opposite angles is 180° ;

that is, when the quadrilateral can be inscribed in a circle.

8. $23 + 29 + 37 + 41 = 130$;

$$\begin{aligned} \therefore \text{maximum area} &= \sqrt{(65 - 23)(65 - 29)(65 - 37)(65 - 41)} \text{ sq. inches} \\ &= 6 \times 6 \times 4 \times 7 \text{ sq. inches} = 7 \text{ sq. feet.} \end{aligned}$$

9. We have $\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$; [Art. 233]

$$\begin{aligned} \therefore \tan^2 \frac{B}{2} &= \frac{1 - \cos B}{1 + \cos B} = \frac{(c + d)^2 - (a - b)^2}{(a + b)^2 - (c - d)^2} \\ &= \frac{(a + c + d - b)(c + d + b - a)}{(a + b + c - d)(a + b + d - c)} = \frac{(\sigma - a)(\sigma - b)}{(\sigma - c)(\sigma - d)}. \end{aligned}$$

10. See figure on page 220. Let $\angle DPA = \beta$;

then

$$a^2 = AP^2 + PB^2 + 2AP \cdot PB \cos \beta;$$

$$c^2 = DP^2 + PC^2 + 2DP \cdot PC \cos \beta;$$

$$b^2 = BP^2 + PC^2 - 2BP \cdot PC \cos \beta;$$

$$d^2 = DP^2 + PA^2 - 2DP \cdot PA \cos \beta;$$

$$\begin{aligned} \therefore (a^2 + c^2) - (b^2 + d^2) &= 2 \cos \beta \{AP(DP + PB) + PC(DP + PB)\} \\ &= 2AC \cdot BD \cos \beta = 2fg \cos \beta. \end{aligned}$$

$$11. \text{ Area} = \frac{1}{2} AC \cdot BD \sin \beta \quad [\text{Art. 231}]$$

$$= \frac{1}{4} \{ (a^2 + c^2) - (b^2 + d^2) \} \tan \beta. \quad [\text{Ex. 10.}]$$

$$12. \text{ We have } a + c = b + d; \therefore (a - d)^2 = (b - c)^2.$$

$$\text{Also } a^2 + d^2 - 2ad \cos A = b^2 + c^2 - 2bc \cos C;$$

$$\therefore \text{ by subtraction, } ad(1 - \cos A) = bc(1 - \cos C);$$

$$\text{that is, } ad - bc = ad \cos A - bc \cos C;$$

$$\therefore a^2 d^2 (1 - \cos^2 A) + b^2 c^2 (1 - \cos^2 C) = 2abcd - 2abcd \cos A \cos C;$$

$$\text{or } a^2 d^2 \sin^2 A + b^2 c^2 \sin^2 C = 2abcd - 2abcd \cos A \cos C.$$

$$\therefore (2S)^2 - 2abcd \sin A \sin C = 2abcd - 2abcd \cos A \cos C.$$

$$\therefore 4S^2 = 2abcd (1 - \cos \overline{A + C}) = 4abcd \sin^2 \frac{A + C}{2}.$$

$$\therefore S = \sqrt{abcd} \sin \frac{A + C}{2}.$$

13. If β be the angle between the diagonals, we have

$$S = \frac{1}{2} fg \sin \beta.$$

$$\therefore f^2 g^2 - 4S^2 = f^2 g^2 \cos^2 \beta$$

$$= \frac{1}{4} \{ (a^2 + c^2) - (b^2 + d^2) \}^2 \quad [\text{Ex. 10}]$$

$$= \frac{1}{4} (2bd - 2ac)^2, \text{ since } a + c = b + d;$$

$$\text{that is, } 4S^2 = f^2 g^2 - (ac - bd)^2.$$

14. (1) By Euc. VI. D, we have

$$(ac + bd) \sin \beta = fg \sin \beta = 2S = (ad + bc) \sin A.$$

$$(2) \cos \beta = \frac{(a^2 + c^2) - (b^2 + d^2)}{2fg} \quad [\text{Ex. 10}]$$

$$= \frac{(a^2 + c^2) - (b^2 + d^2)}{2(ac + bd)}.$$

$$\begin{aligned}
 (3) \quad \tan^2 \frac{\beta}{2} &= \frac{1 - \cos \beta}{1 + \cos \beta} = \frac{(b+d)^2 - (c-a)^2}{(c+a)^2 - (b-d)^2}, \text{ or } \frac{(c+a)^2 - (b-d)^2}{(b+d)^2 - (c-a)^2} \\
 &= \frac{(b+d+c-a)(b+d-c+a)}{(c+a-b+d)(c+a+b-d)}, \text{ or } \frac{(c+a-b+d)(c+a+b-d)}{(b+d+c-a)(b+d-c+a)} \\
 &= \frac{(\sigma-a)(\sigma-c)}{(\sigma-b)(\sigma-d)}, \text{ or } \frac{(\sigma-b)(\sigma-d)}{(\sigma-a)(\sigma-c)}.
 \end{aligned}$$

$$\begin{aligned}
 15. \quad S &= \frac{1}{2}fg \sin \beta \\
 &= \frac{1}{4} \sqrt{4f^2g^2 - 4f^2g^2 \cos^2 \beta} \\
 &= \frac{1}{4} \sqrt{4f^2g^2 - (a^2 + c^2 - b^2 - d^2)^2}. \quad [\text{Ex. 10.}]
 \end{aligned}$$

16. See figure on page 220.

$$\begin{aligned}
 \text{We have} \quad \frac{AP}{PC} &= \frac{\triangle APB}{\triangle CPB} = \frac{\triangle APD}{\triangle CPD} = \frac{\triangle DAB}{\triangle DCB} = \frac{ad}{bc}; \\
 \therefore AP &= \frac{ad}{ad+bc} \cdot AC; \\
 PC &= \frac{bc}{ad+bc} \cdot AC; \\
 \therefore AP \cdot PC &= \frac{abcd}{(ad+bc)^2} \cdot AC^2 = \frac{abcd}{(ad+bc)^2} \cdot \frac{(ad+bc)(ac+bd)}{ab+cd} \\
 &= \frac{abcd(ac+bd)}{(ab+cd)(ad+bc)}.
 \end{aligned}$$

EXAMPLES. XVIII. e. PAGE 225.

1. $242 + 1212 + 1450 = 2904;$
 $\therefore \text{area} = \sqrt{1452 \times 1210 \times 240 \times 2} \text{ sq. yds.}$
 $= 8 \times 3 \times 121 \times 10 \text{ sq. yds.}$
 $= \frac{8 \times 3 \times 121 \times 10}{4840} = 6 \text{ acres.}$
2. $\text{Area} = \frac{200^2 \sin 22\frac{1}{2}^\circ \sin 67\frac{1}{2}^\circ}{2 \sin (22\frac{1}{2}^\circ + 67\frac{1}{2}^\circ)} = \frac{200^2 \cos 45^\circ}{4}$
 $= \frac{10000\sqrt{2}}{2} = \frac{14142}{2} = 7071 \text{ sq. yds.}$

3. We have $\frac{\Delta}{s-a} = \frac{2\Delta}{s-b} = \frac{2\Delta}{s-c}$;

$$\therefore b=c, \text{ and } s-b=2(s-a);$$

$$\therefore \frac{c+a-b}{2} = b+c-a;$$

that is, $\frac{a}{2} = 2b-a;$

that is, $3a=4b.$

4. If a, b, c are in A.P., we have $a+c=2b$;

$$\therefore \frac{1}{r_1} + \frac{1}{r_3} = \frac{s-a+s-c}{\Delta} = \frac{2(s-b)}{\Delta} = \frac{2}{r_2};$$

$$\therefore r_1, r_2, r_3 \text{ are in H.P.}$$

5. We have $s = \frac{z}{x} + \frac{x}{y} + \frac{y}{z}.$

$$\therefore \text{area} = \sqrt{\left(\frac{z}{x} + \frac{x}{y} + \frac{y}{z}\right) \frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x}} = \sqrt{\frac{x}{y} + \frac{y}{z} + \frac{z}{x}}.$$

6. We have $\frac{\sin(A-B)}{\sin(B-C)} = \frac{\sin A}{\sin C} = \frac{\sin(B+C)}{\sin(A+B)};$

$$\therefore \sin(A-B) \sin(A+B) = \sin(B-C) \sin(B+C);$$

or $\sin^2 A - \sin^2 B = \sin^2 B - \sin^2 C;$

$$\therefore a^2 - b^2 = b^2 - c^2;$$

that is, a^2, b^2, c^2 are in A.P.

7. First side $= \frac{a \sin A + b \sin B + c \sin C}{\sin A + \sin B + \sin C} = \frac{a^2 + b^2 + c^2}{a+b+c} = \frac{a^2 + b^2 + c^2}{2s}.$

8. First side $= \frac{a}{\sin A} \cdot (a+b+c) \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
 $= \frac{abc}{2\Delta} \cdot 2s \frac{(s-a)(s-b)(s-c)}{abc} = \frac{\Delta^2}{\Delta} = \Delta.$

9. By Ex. 20, XVIII. a. we have

$$\text{first side} = abc \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

$$= abc \sqrt{\frac{s(s-a)}{(s-b)(s-c)} \cdot \frac{s(s-b)}{(s-c)(s-a)} \cdot \frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{abc s^2}{\Delta} = 4Rs^2 = 4R(r_2 r_3 + r_3 r_1 + r_1 r_2). \quad [\text{XVIII. a. Ex. 24.}]$$

$$10. \quad \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = \frac{r_1}{s} + \frac{r_2}{s} + \frac{r_3}{s} = \frac{r_1 + r_2 + r_3}{s}$$

$$= \frac{r_1 + r_2 + r_3}{(r_2 r_3 + r_3 r_1 + r_1 r_2)^{\frac{1}{2}}}. \quad [\text{XVIII. a. Ex. 24.}]$$

$$11. \quad \text{First side} = bc \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + ca \sqrt{\frac{s(s-b)}{(s-c)(s-a)}} \\ + ab \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{s}{\Delta} \{bc(s-a) + ca(s-b) + ab(s-c)\}$$

$$= \frac{s}{\Delta} \{(bc + ca + ab)s - 3abc\}$$

$$= \frac{abc s^2}{\Delta} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{3}{s} \right)$$

$$= 4Rs^2 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{3}{s} \right).$$

$$12. \quad \text{First side} = \left(\frac{s + s - a + s - b + s - c}{\Delta} \right)^2 = \left(\frac{2s}{\Delta} \right)^2$$

$$= \frac{4s}{\Delta} \cdot \left(\frac{s - a + s - b + s - c}{\Delta} \right) = \frac{4}{r} \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right).$$

$$13. \quad \text{We have } \frac{\Delta}{s} = 6, \quad 2s = 70; \quad \therefore \Delta = 6 \times 35.$$

Let a, b be the sides containing the right angle;

$$\text{then} \quad \frac{1}{2}ab = \Delta = 6 \times 35; \quad \therefore ab = 12 \times 35;$$

$$\text{also} \quad a^2 + b^2 = c^2 = \{70 - (a+b)\}^2;$$

$$\therefore 0 = 70^2 - 140(a+b) + 2ab = 70^2 - 140(a+b) + 12 \times 70;$$

$$\therefore a + b = 41 = 20 + 21,$$

$$\text{and} \quad ab = 12 \times 35 = 20 \times 21;$$

$\therefore$ the two sides containing the right angle are 20, 21, and the hypotenuse
 $= 70 - 41 = 29.$

$$14. \quad \text{It is easily seen from a figure that } \frac{a}{2f} = \tan A;$$

$$\therefore \frac{a}{f} + \frac{b}{g} + \frac{c}{h} = 2(\tan A + \tan B + \tan C)$$

$$= 2 \tan A \tan B \tan C = \frac{abc}{4fgh}.$$

15. Let r, r' be the radii of the inscribed circles; then since the perimeters of the triangle and hexagon are equal we have

$$6r \tan \frac{\pi}{3} = 12r' \tan \frac{\pi}{6};$$

whence

$$\begin{aligned} 3r &= 2r'; \\ \therefore \frac{\pi r^2}{\pi r'^2} &= \frac{4}{9}; \end{aligned}$$

$\therefore$ areas of inscribed circles are as 4 to 9.

$$\begin{aligned} 16. \text{ Perimeter} &= R (\sin 2A + \sin 2B + \sin 2C) = 4R \sin A \sin B \sin C \\ &= 4R \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{abc}{2R^2}. \end{aligned}$$

$$\begin{aligned} 17. \text{ Area} &= \frac{1}{2} 4R \cos \frac{B}{2} 4R \cos \frac{C}{2} \sin \left(90^\circ - \frac{A}{2} \right) \\ &= 8R^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \\ &= 2R^2 (\sin A + \sin B + \sin C) \\ &= R(a + b + c) = \frac{abc(a + b + c)}{4\Delta}. \end{aligned}$$

18. Let O_1, O_2 be the two circumcentres; then O_1O_2 is at right angles to AC at its middle point. Draw O_1N_1, O_2N_2 perpendicular to AB_1 ;

then

$$O_1O_2 = N_1N_2 \operatorname{cosec} A = \frac{c_1 \sim c_2}{2 \sin A}.$$

19. Let f, g be the diagonals; then by Euc. vi. D.,

$$(ac + bd) \sin \beta = fg \sin \beta = 2S.$$

[Art. 231.]

$$\therefore \sin \beta = \frac{2S}{ac + bd}.$$

20. We have $r \cdot II_1 = r \cdot IC \operatorname{cosec} II_1C = ICr \operatorname{cosec} \frac{B}{2} = IC \cdot IB.$

$$\therefore r^3 II_1 \cdot II_2 \cdot II_3 = IA^2 \cdot IB^2 \cdot IC^2.$$

21. Sum of squares of sides of ex-central Δ

$$= 16R^2 \left(\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \right)$$

$$= 16R^2 \left(2 + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right)$$

[XII. d. Ex. 13]

$$= 32R^2 + 32R^2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= 32R^2 + 8Rr = 8R(4R + r).$$

22. Since a circle can be inscribed in the quadrilateral,

$$\therefore a + c = b + d;$$

and since a circle can be circumscribed about the quadrilateral,

$$\begin{aligned}\therefore \cos \beta &= \frac{(a^2 + c^2) \sim (b^2 + d^2)}{2 (ac + bd)} & [\text{XVIII. d. Ex. 14}] \\ &= \frac{(a + c)^2 \sim (b + d)^2 + 2 (ac \sim bd)}{2 (ac + bd)} \\ &= \frac{ac \sim bd}{ac + bd}.\end{aligned}$$

23. (1) We have $2l^2 + 2 \cdot \frac{a^2}{4} = b^2 + c^2;$

$$2m^2 + 2 \cdot \frac{b^2}{4} = c^2 + a^2;$$

$$2n^2 + 2 \cdot \frac{c^2}{4} = a^2 + b^2;$$

$\therefore$ by addition, $4 (l^2 + m^2 + n^2) = 3 (a^2 + b^2 + c^2).$

$$\begin{aligned}(2) \text{ First side} &= \frac{1}{4} \{ (b^2 - c^2) (2b^2 + 2c^2 - a^2) + \text{two similar terms} \} \\ &= \frac{1}{4} [\{ 2 (b^4 - c^4) - a^2 (b^2 - c^2) \} + \dots + \dots] \\ &= 0.\end{aligned}$$

$$\begin{aligned}(3) \quad 4l^2 &= 2b^2 + 2c^2 - a^2; \\ \therefore 16 (l^4 + m^4 + n^4) &= (2b^2 + 2c^2 - a^2)^2 + \dots + \dots \\ &= 9 (a^4 + b^4 + c^4), \text{ on reduction.}\end{aligned}$$

24. We have $r_1 + r_2 + r_3 = 4R + r$ [XVIII. a. Ex. 25],

$$r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$$
 [XVIII. a. Ex. 24],

$$r_1 r_2 r_3 = r s^2;$$
 [XVIII. a. Ex. 15]

$\therefore$ the equation whose roots are r_1, r_2, r_3 is

$$x^3 - (4R + r) x^2 + s^2 x - r s^2 = 0.$$

25. Let PQ be the tangent to inscribed circle parallel to BC , and draw AHD at right angles to PQ and BC ;

then
$$\begin{aligned}\frac{\Delta_1}{\Delta} &= \frac{AP \cdot AQ}{AB \cdot AC} = \frac{AH}{AD} \cdot \frac{AH}{AD} = \left(\frac{AD - 2r}{AD} \right)^2 \\ &= \left(1 - \frac{2\Delta}{s} \cdot \frac{a}{2\Delta} \right)^2 = \frac{(s - a)^2}{s^2}.\end{aligned}$$

$$\begin{aligned}\therefore \frac{\Delta_1}{(s - a)^2} &= \frac{\Delta}{s^2} \\ &= \frac{\Delta_2}{(s - b)^2} = \frac{\Delta_3}{(s - c)^2}, \text{ by symmetry.}\end{aligned}$$

Otherwise. Let the sides and perimeter of triangle APQ be denoted by $a_1, b_1, c_1, 2s_1$; then, by Art. 213, $s_1 = s - a$.

But
$$\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1} = \frac{s}{s_1}; \therefore \frac{\Delta_1}{s_1^2} = \frac{\Delta}{s^2}. \quad [\text{Euc. vi. 19}].$$

That is,
$$\frac{\Delta_1}{(s-a)^2} = \frac{\Delta}{s^2}.$$

26. MN is perpendicular to the bisector of the angle A ;

$\therefore MN$ is parallel to I_2I_3 ;

thus the sides of the $\triangle LMN$ are parallel to the sides of the excentral $\triangle$ and the triangles are similar.

Also
$$MN = 2(s-a) \sin \frac{A}{2};$$

$$\begin{aligned} \therefore \frac{\triangle LMN}{\triangle I_1I_2I_3} &= \frac{MN \cdot NL}{I_2I_3 \cdot I_3I_1} = \frac{2(s-a) \sin \frac{A}{2} \cdot 2(s-b) \sin \frac{B}{2}}{4R \cos \frac{A}{2} \cdot 4R \cos \frac{B}{2}} \\ &= \frac{(s-a)(s-b)(s-c)}{4R^2s} = \frac{\Delta^2}{4R^2s^2} = \frac{r^2}{4R^2}. \end{aligned}$$

27. We have $\angle PBC = \angle PCB = A$; $\therefore \angle QPR = 180^\circ - 2A$.

Similarly $\angle PQR = 180^\circ - 2B$, $\angle QRP = 180^\circ - 2C$.

Again
$$AQ = \frac{AC \sin B}{\sin AQC} = \frac{b \sin B}{\sin 2B} = \frac{b}{2 \cos B};$$

similarly
$$AR = \frac{c}{2 \cos C}.$$

$$\therefore QR = \frac{b}{2 \cos B} + \frac{c}{2 \cos C} = \frac{b \cos C + c \cos B}{2 \cos B \cos C} = \frac{a}{2 \cos B \cos C}.$$

28. (1) We have $pc \sin \frac{A}{2} + pb \sin \frac{A}{2} = 2\Delta = bc \sin A$;

$$\therefore p(b+c) = 2bc \cos \frac{A}{2};$$

$$\therefore \frac{1}{p} \cos \frac{A}{2} = \frac{b+c}{2bc} = \frac{1}{2} \left(\frac{1}{b} + \frac{1}{c} \right);$$

similarly
$$\frac{1}{q} \cos \frac{B}{2} = \frac{1}{2} \left(\frac{1}{c} + \frac{1}{a} \right), \quad \frac{1}{r} \cos \frac{C}{2} = \frac{1}{2} \left(\frac{1}{a} + \frac{1}{b} \right).$$

$$\therefore \frac{1}{p} \cos \frac{A}{2} + \frac{1}{q} \cos \frac{B}{2} + \frac{1}{r} \cos \frac{C}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

$$\begin{aligned}
 (2) \quad \text{We have } pqr &= \frac{8a^2b^2c^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{(b+c)(c+a)(a+b)} \\
 &= \frac{2a^2b^2c^2 (\sin A + \sin B + \sin C)}{(b+c)(c+a)(a+b)} \\
 &= \frac{4\Delta abc (a+b+c)}{(b+c)(c+a)(a+b)}; \\
 \therefore \frac{pqr}{4\Delta} &= \frac{abc (a+b+c)}{(b+c)(c+a)(a+b)}.
 \end{aligned}$$

29. Let O be the orthocentre; then G, H, K are the middle points of OL, OM, ON respectively; therefore the sides of the $\triangle LMN$ are double the sides of the pedal triangle and parallel to them.

Therefore also the angles are equal to the angles of the pedal triangle.

$$\begin{aligned}
 (1) \quad \text{Area of } \triangle LMN &= \frac{1}{2} \cdot 2a \cos A \cdot 2b \cos B \cdot \sin (180^\circ - 2C) \\
 &= 4ab \sin C \cdot \cos A \cos B \cos C = 8\Delta \cos A \cos B \cos C.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad AL &= AG + OG = c \sin B + c \cos B \cot C \\
 &= \frac{c \cos (B - C)}{\sin C} = \frac{a \cos (B - C)}{\sin A};
 \end{aligned}$$

$$\begin{aligned}
 \therefore AL \sin A + BM \sin B + CN \sin C &= a \cos (B - C) + b \cos (C - A) + c \cos (A - B) \\
 &= 2R \{ \sin \overline{B + C} \cos \overline{B - C} + \dots + \dots \} \\
 &= R [(\sin 2B + \sin 2C) + \dots + \dots] \\
 &= 2R [\sin 2A + \sin 2B + \sin 2C] \\
 &= 8R \sin A \sin B \sin C.
 \end{aligned}$$

30. Let P be the centre of the circle inscribed between the in-circle and the sides AB, AC ; then

$$\begin{aligned}
 (1) \quad \frac{r - r_a}{r + r_a} &= \sin \frac{A}{2}; \\
 \therefore r_a &= r \cdot \frac{1 - \sin \frac{A}{2}}{1 + \sin \frac{A}{2}} = r \tan^2 \frac{\pi - A}{4}. \quad [\text{Compare XI. f. Ex. 15.}]
 \end{aligned}$$

$$(1) \quad \text{Also} \quad \frac{\pi - A}{4} + \frac{\pi - B}{4} + \frac{\pi - C}{4} = \frac{\pi}{2};$$

$$\begin{aligned}
 \therefore \sqrt{r_b r_c} + \sqrt{r_c r_a} + \sqrt{r_a r_b} &= r \left(\tan \frac{\pi - B}{4} \tan \frac{\pi - C}{4} + \dots + \dots \right) \\
 &= r.
 \end{aligned}$$

31. It is easily seen that the triangle XYZ is the same as the triangle PQR in Ex. 27.

$$\begin{aligned}\therefore \text{Perimeter} &= \frac{a}{2 \cos B \cos C} + \frac{b}{2 \cos C \cos A} + \frac{c}{2 \cos A \cos B} \\ &= \frac{R (\sin 2A + \sin 2B + \sin 2C)}{2 \cos A \cos B \cos C} = 2R \tan A \tan B \tan C.\end{aligned}$$

$$\begin{aligned}\text{Area} &= \frac{1}{2} \cdot \frac{a}{2 \cos B \cos C} \cdot \frac{b}{2 \cos C \cos A} \cdot \sin 2C \\ &= \frac{ab \sin C}{4 \cos A \cos B \cos C} = R^2 \tan A \tan B \tan C.\end{aligned}$$

32. Let XYZ be the triangle formed by the tangents, and let the perpendiculars from X, Y, Z to the chord be represented by x, y, z respectively.

Then the area of

$$\begin{aligned}\Delta XYZ &= \Delta BXC - \Delta ACY + \Delta AZB \\ &= \frac{1}{2} x \cdot BC - \frac{1}{2} y \cdot AC + \frac{1}{2} z \cdot AB.\end{aligned}$$

Now $\frac{x}{CX} = \frac{DC}{CO}$, since OCX is a right angle.

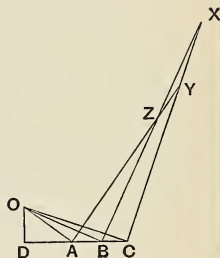
Also $\frac{CX}{CO} = \frac{DB}{OD} = \frac{DB}{p}$, since O, B, C, X are concyclic.

$$\therefore x = \frac{DC \cdot DB}{p}.$$

$\therefore \Delta BXC = \frac{1}{2p} \cdot DC \cdot DB (DC - DB) = \frac{cb(c-b)}{2p}$, if a, b, c denote the distances of A, B, C from D .

$$\begin{aligned}\therefore \text{area of } \Delta XYZ &= \frac{cb(c-b) - ca(c-a) + ba(b-a)}{2p} \\ &= \frac{cb(c-b) + ba(b-a) + ac(a-c)}{2p} \\ &= -\frac{(c-b)(b-a)(a-c)}{2p} = \frac{BC \cdot CA \cdot AB}{2p},\end{aligned}$$

for $CA = c - a = -(a - c)$.



MISCELLANEOUS EXAMPLES. F. PAGE 228.

1. We have $\cos(\alpha + \beta) = \cos(180^\circ - \gamma + \delta) = -\cos(\gamma + \delta)$;
 $\therefore \cos \alpha \cos \beta - \sin \alpha \sin \beta = \sin \gamma \sin \delta - \cos \gamma \cos \delta$;
 $\therefore \cos \alpha \cos \beta + \cos \gamma \cos \delta = \sin \alpha \sin \beta + \sin \gamma \sin \delta$.

$$\begin{aligned} 2. \text{ First side} &= \frac{\cos(15^\circ - A) \sin 15^\circ - \sin(15^\circ - A) \cos 15^\circ}{\sin 15^\circ \cos 15^\circ} \\ &= \frac{2 \sin A}{\sin 30^\circ} = 4 \sin A. \end{aligned}$$

$$\begin{aligned} 3. \cot A + \sin A \operatorname{cosec} B \operatorname{cosec} C &= \frac{\cos A \sin B \sin C + 1 - \cos^2 A}{\sin A \sin B \sin C} \\ &= \frac{\cos A \{\sin B \sin C + \cos(B + C)\} + 1}{\sin A \sin B \sin C} \\ &= \frac{\cos A \cos B \cos C + 1}{\sin A \sin B \sin C}, \end{aligned}$$

which is symmetrical with respect to A, B, C .

$$\begin{aligned} 4. \text{ We have } \sin B &= \frac{b \sin A}{a} = \frac{\sqrt{8} \cdot \sin 30^\circ}{2} = \frac{1}{\sqrt{2}}; \\ \therefore B &= 45^\circ, \text{ or } 135^\circ; & [\text{Art. 148. (iii)}]; \\ \therefore C &= 105^\circ, \text{ or } 15^\circ; \\ \therefore c &= \frac{a \sin C}{\sin A} = 4 \cos 15^\circ, \text{ or } 4 \sin 15^\circ; \\ \therefore c &= \sqrt{6} + \sqrt{2}, \text{ or } \sqrt{6} - \sqrt{2}. \end{aligned}$$

$$\begin{aligned} 5. (1) \text{ We have } \cot 18^\circ \tan 36^\circ &= \frac{2 \cos 18^\circ \sin 36^\circ}{2 \sin 18^\circ \cos 36^\circ} \\ &= \frac{\sin 54^\circ + \sin 18^\circ}{\sin 54^\circ - \sin 18^\circ} \\ &= \frac{\sqrt{5} + 1 + (\sqrt{5} - 1)}{\sqrt{5} + 1 - (\sqrt{5} - 1)} = \sqrt{5}. \end{aligned}$$

$$(2) \sin 36^\circ = \sin 144^\circ, \text{ and } \sin 72^\circ = \sin 108^\circ;$$

$$\begin{aligned} \therefore \text{ first side} &= 4 (2 \sin 72^\circ \sin 36^\circ)^2 = 4 (\cos 36^\circ - \cos 108^\circ)^2 \\ &= 4 (\cos 36^\circ + \sin 18^\circ)^2 = 4 \left(\frac{\sqrt{5} + 1}{4} + \frac{\sqrt{5} - 1}{4} \right)^2 \\ &= 5. \end{aligned}$$

$$6. \quad \log 2 = \cdot 30103; \therefore \log 4 = \cdot 60206$$

$$\log 3 = \cdot 47712; \therefore \log 9 = \cdot 95424$$

$$\log 11 = 1\cdot 04139$$

$$\overline{2\cdot 59769}$$

$$\therefore \log \cdot 0396 = \overline{2\cdot 59769};$$

$$\therefore \log (\cdot 0396)^{90} = -180 + 53\cdot 7921 = \overline{127\cdot 7921};$$

$\therefore$ number of ciphers before the first significant digit in $(\cdot 0396)^{90}$ is 126.

7. Let P, Q be the two positions of the observer;

then $\angle QPB = 30^\circ, \angle QBP = 45^\circ, PQ = 50$ yards;

$$\therefore PB = \frac{50 \sin 75^\circ}{\sin 45^\circ} = \frac{50 \sqrt{2} (\sqrt{3} + 1)}{2 \sqrt{2}} = 25 (\sqrt{3} + 1) = 68\cdot 3 \text{ yds.},$$

$$QB = \frac{50 \sin 30^\circ}{\sin 45^\circ} = 50 \sqrt{2} \cdot \frac{1}{2} = 35\cdot 35 \text{ yds.}$$

$$8. \quad \text{First side} = 2 + \frac{1}{2} \{ \cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2(\alpha + \beta + \gamma) \}$$

$$= 2 + \cos(\alpha + \beta) \cos(\alpha - \beta) + \cos(\alpha + \beta + 2\gamma) \cos(\alpha + \beta)$$

$$= 2 + 2 \cos(\alpha + \beta) \cos(\beta + \gamma) \cos(\gamma + \alpha).$$

$$9. \quad (1) \quad \tan 40^\circ + \cot 40^\circ = \frac{\sin^2 40^\circ + \cos^2 40^\circ}{\cos 40^\circ \sin 40^\circ} = \frac{2}{2 \sin 40^\circ \cos 40^\circ}$$

$$= \frac{2}{\sin 80^\circ} = 2 \sec 10^\circ.$$

$$(2) \quad \tan 70^\circ + \tan 20^\circ = \tan 20^\circ + \cot 20^\circ = \frac{2}{\sin 40^\circ}, \text{ as in (1),}$$

$$= 2 \operatorname{cosec} 40^\circ.$$

$$10. \quad (1) \quad \text{First side} = 2 \sin 4a - 2 \cos 6a \sin 4a$$

$$= 2 \sin 4a (1 - \cos 6a) = 4 \sin 4a \sin^2 3a$$

$$= 16 \sin a \cos a \cos 2a \sin^2 3a.$$

$$(2) \quad \text{First side} = \sin \frac{2\pi}{7} - 2 \cos \frac{5\pi}{7} \sin \frac{\pi}{7}$$

$$= 2 \sin \frac{\pi}{7} \left(\cos \frac{\pi}{7} - \cos \frac{5\pi}{7} \right)$$

$$= 4 \sin \frac{\pi}{7} \sin \frac{2\pi}{7} \sin \frac{3\pi}{7}$$

$$= 4 \sin \frac{\pi}{7} \sin \frac{5\pi}{7} \sin \frac{3\pi}{7}.$$

11. We have $\sin C = \frac{c \sin B}{b} = \frac{6 - 2\sqrt{3}}{2(3\sqrt{2} - \sqrt{6})} = \frac{1}{\sqrt{2}}$;
 $\therefore C = 45^\circ$ or 135° ; [Art. 148, (iii)];

$$\therefore A = 105^\circ \text{ or } 15^\circ;$$

$$\therefore a = \frac{c \sin A}{\sin C} = \frac{(6 - 2\sqrt{3})\sqrt{2}(\sqrt{3} + 1)}{2\sqrt{2}}, \text{ or } \frac{(6 - 2\sqrt{3})\sqrt{2}(\sqrt{3} - 1)}{2\sqrt{2}};$$

that is, $a = 2\sqrt{3}$, or $4\sqrt{3} - 6$.

12. Let C be the rock and A, B the two positions of the ship.

Then we have $\angle BAC = \angle BCA = 67\frac{1}{2}^\circ$;

$$\therefore BC = BA = 10 \text{ miles,}$$

and $AC = 2AB \sin 22\frac{1}{2}^\circ = 10\sqrt{2} - \sqrt{2} \text{ miles.}$ [Art. 251.]

13. First side
$$= \frac{\sin^2 B - \sin^2 C}{\cos B + \cos C} + \dots + \dots$$

$$= \frac{\cos^2 C - \cos^2 B}{\cos B + \cos C} + \dots + \dots$$

$$= (\cos C - \cos B) + \dots + \dots$$

$$= 0.$$

14. We have
$$\frac{1}{\cos(\theta - \alpha)} + \frac{1}{\cos(\theta + \alpha)} = \frac{2}{\cos \theta};$$

$$\therefore \frac{4 \cos \theta \cos \alpha}{\cos 2\theta + \cos 2\alpha} = \frac{2}{\cos \theta};$$

$$\therefore 2 \cos^2 \theta \cos \alpha = 2 \cos^2 \theta - 1 + 2 \cos^2 \alpha - 1;$$

$$\therefore \cos^2 \theta (\cos \alpha - 1) = \cos^2 \alpha - 1;$$

$$\therefore \cos^2 \theta = \cos \alpha + 1 = 2 \cos^2 \frac{\alpha}{2};$$

whence
$$\cos \theta = \sqrt{2} \cos \frac{\alpha}{2}.$$

15. We have
$$\sin \alpha \cos \alpha = \sin^2 \beta$$

$$= \frac{1}{2}(1 - \cos 2\beta);$$

$$\therefore \cos 2\beta = 1 - 2 \sin \alpha \cos \alpha = (\cos \alpha - \sin \alpha)^2$$

$$= 2 \cos^2 \left(\frac{\pi}{4} + \alpha \right).$$

16. See figure of Art. 223.

$$OG = BG \cot BOG = BG \cot C = c \cos B \cot C$$

$$= 2R \cos B \cos C;$$

similarly $OH = 2R \cos C \cos A, \quad OK = 2R \cos A \cos B.$

17. We have $\cos \theta = \frac{\cos u - e}{1 - e \cos u}$.

$$\therefore \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{1 - e \cos u - \cos u + e}{1 - e \cos u + \cos u - e} = \frac{(1 + e)(1 - \cos u)}{(1 - e)(1 + \cos u)};$$

$$\therefore \tan \frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \cdot \tan \frac{u}{2}.$$

18. The sum of the squares on the sides containing the right angle

$$\begin{aligned} &= 4(1 + \sin \theta)^2 + 4(1 + \cos \theta)^2 + \cos^2 \theta + \sin^2 \theta + 4 \cos \theta (1 + \sin \theta) + 4 \sin \theta (1 + \cos \theta) \\ &= 8 + 8(\sin \theta + \cos \theta) + 4 + 1 + 4(\sin \theta + \cos \theta) + 8 \sin \theta \cos \theta \\ &= 9 + 12(\sin \theta + \cos \theta) + 4(1 + 2 \sin \theta \cos \theta) \\ &= 9 + 12(\sin \theta + \cos \theta) + 4(\sin \theta + \cos \theta)^2 \\ &= \{3 + 2(\sin \theta + \cos \theta)\}^2. \end{aligned}$$

$$\therefore \text{hypotenuse} = 3 + 2(\sin \theta + \cos \theta).$$

19. In the figure of Art. 227 let O be the centre of in-circle of $I_1 I_2 I_3$, and O_1 the centre of the ex-circle opposite to I_1 . Let R' be the circum-radius of $I_1 I_2 I_3$.

Then, as in Art. 220, $OO_1 = 4R' \sin \frac{I_2 I_1 I_3}{2}$;

but $\angle I_2 I_1 I_3 = \frac{\pi}{2} - \frac{A}{2}$; and $R' = 2R$; [Arts. 221, 222];

$$\therefore OO_1 = 8R \sin \left(\frac{\pi}{4} - \frac{A}{4} \right) = 8R \sin \frac{B+C}{4}.$$

20. The sides of the ex-central triangle of the triangle $I_1 I_2 I_3$ are

$$4R' \cos \frac{\pi - A}{4}, \quad 4R' \cos \frac{\pi - B}{4}, \quad 4R' \cos \frac{\pi - C}{4}, \quad [\text{Art. 221}],$$

that is, $8R \cos \frac{B+C}{4}, \quad 8R \cos \frac{C+A}{4}, \quad 8R \cos \frac{A+B}{4}.$

21. We have

$$(1 + \cos \alpha)(1 + \cos \beta)(1 + \cos \gamma) = (1 - \cos \alpha)(1 - \cos \beta)(1 - \cos \gamma);$$

$$\therefore \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} = \pm \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2};$$

$$\therefore \text{each expression} = 8 \cos^2 \frac{\alpha}{2} \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2}$$

$$= \pm 8 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \cdot \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$$

$$= \pm \sin \alpha \sin \beta \sin \gamma.$$

22. Let $\alpha, \beta, \gamma, \delta$ be four angles such that $\alpha + \beta + \gamma + \delta = 180^\circ$;

then $\cos(\alpha + \beta) = \cos(180^\circ - \gamma + \delta) = -\cos(\gamma + \delta)$;

$$\therefore \cos \alpha \cos \beta + \cos \gamma \cos \delta = \sin \alpha \sin \beta + \sin \gamma \sin \delta;$$

similarly $\cos \beta \cos \gamma + \cos \delta \cos \alpha = \sin \beta \sin \gamma + \sin \delta \sin \alpha$;

$$\cos \gamma \cos \alpha + \cos \beta \cos \delta = \sin \gamma \sin \alpha + \sin \beta \sin \delta;$$

$\therefore$ by addition we have the sum of the products of the cosines taken two together equal to the sum of the products of the sines taken two together.

$$23. (1) II_1 \cdot II_2 \cdot II_3 = 4R \sin \frac{A}{2} \cdot 4R \sin \frac{B}{2} \cdot 4R \sin \frac{C}{2} \quad [\text{Art. 220}]$$

$$= 16R^2 \cdot 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= 16R^2 r.$$

$$(2) II_1^2 + I_2 I_3^2 = 16R^2 \sin^2 \frac{A}{2} + 16R^2 \cos^2 \frac{A}{2} \quad [\text{Arts. 220, 221}]$$

$$= 16R^2.$$

24. (1) Let α, β, γ be the angles;

$$\text{then } \cos \alpha = \frac{\cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} - \cos^2 \frac{A}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{1 + \cos \frac{B+C}{2} \cos \frac{B-C}{2} - \cos^2 \frac{A}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}}$$

$$= \frac{\sin \frac{A}{2} \cos \frac{B-C}{2} + \sin^2 \frac{A}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{2 \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}}$$

$$= \sin \frac{A}{2} = \cos \left(90^\circ - \frac{A}{2} \right).$$

Thus the angles are $90^\circ - \frac{A}{2}$, $90^\circ - \frac{B}{2}$, $90^\circ - \frac{C}{2}$.

(2) Here

$$\cos \alpha = \frac{\sin^2 2B + \sin^2 2C - \sin^2 2A}{2 \sin 2B \sin 2C} = \frac{1 - \cos 2(B+C) \cos 2(B-C) - \sin^2 2A}{2 \sin 2B \sin 2C}$$

$$= \frac{-\cos 2A \cos 2(B-C) + \cos 2A}{2 \sin 2B \sin 2C}$$

$$= \frac{-\cos 2A \{ \cos 2(B-C) - \cos 2(B+C) \}}{2 \sin 2B \sin 2C} = -\cos 2A.$$

Thus the angles are $180^\circ - 2A$, $180^\circ - 2B$, $180^\circ - 2C$.

25. The expression

$$\begin{aligned}
 &= \{\sin(\theta + \alpha) + \sin(\theta + \beta)\}^2 - 2 \sin(\theta + \alpha) \sin(\theta + \beta) \\
 &\quad - 2 \cos(\alpha - \beta) \sin(\theta + \alpha) \sin(\theta + \beta) \\
 &= \left\{ 2 \sin \frac{2\theta + \alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \right\}^2 - 2 \sin(\theta + \alpha) \sin(\theta + \beta) (1 + \cos \alpha - \beta) \\
 &= 4 \cos^2 \frac{\alpha - \beta}{2} \left\{ \sin^2 \frac{2\theta + \alpha + \beta}{2} - \sin(\theta + \alpha) \sin(\theta + \beta) \right\} \\
 &= 2 \cos^2 \frac{\alpha - \beta}{2} \{ 1 - \cos(2\theta + \alpha + \beta) - \cos(\alpha - \beta) + \cos(2\theta + \alpha + \beta) \} \\
 &= 2 \cos^2 \frac{\alpha - \beta}{2} (1 - \cos \alpha - \beta),
 \end{aligned}$$

which is independent of θ .

26. See figure on page 220.

Since the quadrilateral is described about a circle,

$$\therefore a + c = b + d; \text{ that is, } a - d = b - c.$$

$$\text{Now} \quad a^2 + d^2 - 2ad \cos A = BD^2 = b^2 + c^2 - 2bc \cos C;$$

$$\therefore (a - d)^2 + 2ad(1 - \cos A) = (b - c)^2 + 2bc(1 - \cos C);$$

$$\therefore ad \sin^2 \frac{A}{2} = bc \sin^2 \frac{C}{2}.$$

27. Let the tangent parallel to BC meet AC in M ; and let AG , the perpendicular from A to BC , meet the tangent in X ; then

$$\frac{p}{a} = \frac{AM}{AC} = \frac{AX}{AG} = \frac{AG - 2r}{AG}$$

$$= 1 - \frac{ar}{\Delta} = 1 - \frac{a}{s};$$

$$\therefore \frac{p}{a} + \frac{q}{b} + \frac{r}{c} = 1 - \frac{a}{s} + 1 - \frac{b}{s} + 1 - \frac{c}{s}$$

$$= 3 - \frac{2s}{s} = 1.$$

28. Take the figure of Art. 199. Then we have

$$\angle PAB = 13^\circ 14' 12'', \quad \angle PBC = 56^\circ 24' 36'', \quad \angle BPA = 43^\circ 10' 24''.$$

Also $PC = 1566$. Let $AB = x$ ft.

$$\begin{aligned} x &= \frac{PB \sin 43^\circ 10' 24''}{\sin 13^\circ 14' 12''} \\ \log \sin 43^\circ 10' &= \bar{1} \cdot 8351341 \\ \frac{24}{60} \times 1347 &= 539 \\ \log PB &= 3 \cdot 2741376 \\ &\quad \underline{3 \cdot 1093256} \\ \log \sin 13^\circ 14' 12'' &= \bar{1} \cdot 3597858 \\ \log x &= 3 \cdot 7495398 \\ \log 5617 \cdot 4 &= 3 \cdot 7495353 \\ &\quad \underline{45} \\ &\quad 6 \qquad \qquad \underline{46} \end{aligned}$$

$$\begin{aligned} PB &= 1566 \operatorname{cosec} 56^\circ 24' 36''. \\ \log \operatorname{cosec} 56^\circ 24' &= \cdot 0793961 \\ \text{subtract } \frac{36}{60} \times 839 &= 503 \\ &\quad \underline{\cdot 0793458} \\ \log 1566 &= 3 \cdot 1947918 \\ \log PB &= 3 \cdot 2741376 \\ \log \sin 13^\circ 14' &= \bar{1} \cdot 3596785 \\ \frac{12}{60} \times 5369 &= 1073 \\ &\quad \underline{\bar{1} \cdot 3597858} \end{aligned}$$

Thus $x = 5617 \cdot 46$ ft., whence it easily follows that the speed of the train is 21·3 miles per hour.

29. Let A represent the harbour, C the fort, B the position of the ship when 20 miles from C .

Then $AC = 27 \cdot 23$ miles, $CB = 20$ miles, $\angle CAB = 46^\circ 8' 8 \cdot 6''$.

$$\begin{aligned} \sin B &= \frac{27 \cdot 23 \sin 46^\circ 8' 8 \cdot 6''}{20} \\ \log 27 \cdot 23 &= 1 \cdot 4350476 \\ \log \sin 46^\circ 8' &= \bar{1} \cdot 8579078 \\ \frac{86}{600} \times 1215 &= 174 \\ &\quad \underline{1 \cdot 2929728} \\ \log 20 &= 1 \cdot 3010300 \\ \log \sin B &= \bar{1} \cdot 9919428 \\ \log \sin 78^\circ 59' &= \bar{1} \cdot 9919220 \\ &\quad \underline{208} \end{aligned}$$

$$\begin{aligned} \text{Diff. for } 60'' &= 246; \\ \therefore \text{prop}^l. \text{ increase} &= \frac{208}{246} \times 60'' = 50 \cdot 7''. \end{aligned}$$

$\therefore B = 78^\circ 59' 50 \cdot 7''$, or $101^\circ 0' 9 \cdot 3''$, both values being admissible since $a < b$.

Hence with the third figure of page 131 we have

$$\angle ACB_1 = 54^\circ 52' 0 \cdot 7'', \quad \angle ACB_2 = 32^\circ 51' 42 \cdot 1''.$$

In $\triangle ACB_1$,

$$AB_1 = \frac{20 \sin 54^\circ 52' 0.7''}{\sin 46^\circ 8' 8.6''},$$

$$\log \sin 54^\circ 52' = \bar{1}.9126551$$

$$\frac{7}{600} \times 889 = 10$$

$$\log 20 = 1.3010300$$

$$\underline{1.2136861}$$

$$\log \sin 46^\circ 8' 8.6'' = \bar{1}.8579252$$

$$\log AB_1 = 1.3557609$$

$$\log 22.686 = 1.3557579$$

$$\underline{30}$$

$$2 \quad \underline{38}$$

$$\therefore AB_1 = 22.6862 \text{ miles.}$$

In $\triangle ACB_2$,

$$AB_2 = \frac{20 \sin 32^\circ 51' 42.1''}{\sin 46^\circ 8' 8.6''};$$

$$\log \sin 32^\circ 51' = \bar{1}.7343529$$

$$\frac{421}{600} \times 1956 = 1372$$

$$\log 20 = 1.3010300$$

$$\underline{1.0355201}$$

$$\log \sin 46^\circ 8' 8.6'' = \bar{1}.8579252$$

$$\log AB_2 = 1.1775949$$

$$\log 15.052 = 1.1775942$$

$$\underline{38}$$

$$\therefore AB_2 = 15.052 \text{ miles.}$$

Thus the time taken is approximately 2.27 hours or 1.5 hours; that is the ship will be 20 miles from the fort in 2 hrs. 16 min. or in 1 hr. 30 min.

EXAMPLES. XIX. a. PAGE 235.

$$1. \sin \theta = \frac{1}{2} = \sin \frac{\pi}{6};$$

$$\therefore \theta = n\pi + (-1)^n \frac{\pi}{6}.$$

$$3. \cos \theta = \frac{1}{2} = \cos \frac{\pi}{3};$$

$$\therefore \theta = 2n\pi \pm \frac{\pi}{3}.$$

$$5. \cot \theta = -\sqrt{3} = \cot \left(-\frac{\pi}{6} \right);$$

$$\therefore \theta = n\pi - \frac{\pi}{6}.$$

$$7. \cos^2 \theta = \frac{1}{2};$$

$$\therefore \cos \theta = \pm \frac{1}{\sqrt{2}};$$

$$\theta = 2n\pi \pm \frac{\pi}{4}, \text{ or } 2n\pi \pm \left(\pi - \frac{\pi}{4} \right).$$

$$2. \sin \theta = \frac{1}{\sqrt{2}} = \sin \frac{\pi}{4};$$

$$\therefore \theta = n\pi + (-1)^n \frac{\pi}{4}.$$

$$4. \tan \theta = \sqrt{3} = \tan \frac{\pi}{3};$$

$$\therefore \theta = n\pi + \frac{\pi}{3}.$$

$$6. \sec \theta = -\sqrt{2} = \sec \frac{3\pi}{4};$$

$$\therefore \theta = 2n\pi \pm \frac{3\pi}{4}.$$

$$8. \tan^2 \theta = \frac{1}{3};$$

$$\therefore \tan \theta = \pm \frac{1}{\sqrt{3}} = \tan \left(\pm \frac{\pi}{6} \right);$$

$$\therefore \theta = n\pi \pm \frac{\pi}{6}.$$

Both of these are included in $n\pi \pm \frac{\pi}{4}$.

$$9. \operatorname{cosec}^2 \theta = \frac{4}{3};$$

$$\therefore \cot^2 \theta = \frac{1}{3};$$

$$\therefore \cot \theta = \pm \frac{1}{\sqrt{3}} = \cot \left(\pm \frac{\pi}{3} \right);$$

$$\therefore \theta = n\pi \pm \frac{\pi}{3}.$$

$$12. \sec^2 \theta = \sec^2 \alpha;$$

$$\therefore \tan^2 \theta = \tan^2 \alpha;$$

$$\therefore \theta = n\pi \pm \alpha.$$

$$14. \operatorname{cosec} 3\theta = \operatorname{cosec} 3\alpha;$$

$$\therefore 3\theta = n\pi + (-1)^n 3\alpha;$$

$$\therefore \theta = \frac{n\pi}{3} + (-1)^n \alpha.$$

$$16. \sin 5\theta + \sin \theta = \sin 3\theta;$$

$$\therefore 2 \sin 3\theta \cos 2\theta = \sin 3\theta;$$

$$\therefore \sin 3\theta = 0,$$

$$\text{or} \quad \cos 2\theta = \frac{1}{2},$$

$$\text{whence} \quad \theta = \frac{n\pi}{3}, \text{ or } n\pi \pm \frac{\pi}{6}.$$

$$18. \quad \sin 4\theta + \sin 2\theta - (\sin 3\theta + \sin \theta) = 0;$$

$$\therefore 2 \sin 3\theta \cos \theta - 2 \sin 2\theta \cos \theta = 0;$$

$$\therefore 2 \cos \theta \cdot 2 \cos \frac{5\theta}{2} \sin \frac{\theta}{2} = 0;$$

$$\therefore \cos \theta = 0, \text{ or } \cos \frac{5\theta}{2} = 0, \text{ or } \sin \frac{\theta}{2} = 0.$$

$$\therefore \theta = \frac{(2n+1)\pi}{2}, \text{ or } \frac{(2n+1)\pi}{5}, \text{ or } 2n\pi.$$

19. As in Example 18, we obtain

$$4 \cos \theta \cos 4\theta \cos 2\theta = 0;$$

$$\text{whence} \quad \theta = \frac{(2n+1)\pi}{2}, \text{ or } \frac{(2n+1)\pi}{4}, \text{ or } \frac{(2n+1)\pi}{8}.$$

$$10. \cos \theta = \cos \alpha;$$

$$\therefore \theta = 2n\pi \pm \alpha.$$

$$11. \tan^2 \theta = \tan^2 \alpha;$$

$$\therefore \tan \theta = \pm \tan \alpha = \tan (\pm \alpha);$$

$$\therefore \theta = n\pi \pm \alpha.$$

$$13. \tan 2\theta = \tan \theta;$$

$$\therefore 2\theta = n\pi + \theta;$$

$$\therefore \theta = n\pi.$$

$$15. \cos 3\theta = \cos 2\theta;$$

$$\therefore 3\theta = 2n\pi \pm 2\theta;$$

$$\therefore \theta = 2n\pi, \text{ or } \frac{2n\pi}{5}.$$

$$17. \cos \theta - \cos 7\theta = \sin 4\theta;$$

$$\therefore 2 \sin 4\theta \sin 3\theta = \sin 4\theta;$$

$$\therefore \sin 4\theta = 0, \text{ or } \sin 3\theta = \frac{1}{2};$$

$$\text{whence} \quad \theta = \frac{n\pi}{4}, \text{ or } \frac{n\pi}{3} + (-1)^n \frac{\pi}{18}.$$

20.

$$\sin 5\theta \cos \theta = \sin 6\theta \cos 2\theta;$$

$$\therefore \sin 6\theta + \sin 4\theta = \sin 8\theta + \sin 4\theta;$$

$$\therefore \sin 8\theta - \sin 6\theta = 0;$$

$$\therefore 2 \sin \theta \cos 7\theta = 0;$$

$$\therefore \sin \theta = 0, \text{ or } \cos 7\theta = 0;$$

whence

$$\theta = n\pi, \text{ or } \frac{(2n+1)\pi}{14}.$$

21.

$$\sin 11\theta \sin 4\theta + \sin 5\theta \sin 2\theta = 0;$$

$$\therefore \cos 7\theta - \cos 15\theta + \cos 3\theta - \cos 7\theta = 0;$$

$$\therefore 2 \sin 9\theta \sin 6\theta = 0;$$

whence

$$\theta = \frac{n\pi}{9}, \text{ or } \frac{n\pi}{6}.$$

22.

$$\sqrt{2} \cos 3\theta - \cos \theta = \cos 5\theta;$$

$$\therefore \sqrt{2} \cos 3\theta = 2 \cos 3\theta \cos 2\theta;$$

$$\therefore \cos 3\theta = 0, \text{ or } \cos 2\theta = \frac{1}{\sqrt{2}};$$

$$\therefore \theta = \frac{(2n+1)\pi}{6}, \text{ or } n\pi \pm \frac{\pi}{8}.$$

23.

$$\sin 7\theta - \sqrt{3} \cos 4\theta = \sin \theta;$$

$$\therefore 2 \cos 4\theta \sin 3\theta = \sqrt{3} \cos 4\theta;$$

$$\therefore \cos 4\theta = 0, \text{ or } \sin 3\theta = \frac{\sqrt{3}}{2};$$

whence

$$\theta = \frac{(2n+1)\pi}{8}, \text{ or } \frac{n\pi}{3} + (-1)^n \frac{\pi}{9}.$$

24.

$$1 + \cos \theta = 2 \sin^2 \theta;$$

$$\therefore 1 + \cos \theta = 2 - 2 \cos^2 \theta;$$

$$\therefore 2 \cos^2 \theta + \cos \theta - 1 = 0;$$

$$\therefore \cos \theta = -1, \text{ or } \frac{1}{2};$$

$$\therefore \theta = (2n+1)\pi, \text{ or } 2n\pi \pm \frac{\pi}{3}.$$

25.

$$\tan^2 \theta + \sec \theta = 1;$$

$$\therefore \sec^2 \theta + \sec \theta - 2 = 0;$$

$$\therefore \sec \theta = -2, \text{ or } 1;$$

$$\therefore \theta = 2n\pi \pm \frac{2\pi}{3}, \text{ or } 2n\pi.$$

$$\begin{aligned}
 26. \quad & \cot^2 \theta - 1 = \operatorname{cosec} \theta; \\
 & \therefore \operatorname{cosec}^2 \theta - \operatorname{cosec} \theta - 2 = 0; \\
 & \therefore \operatorname{cosec} \theta = 2, \text{ or } -1; \\
 & \therefore \theta = n\pi + (-1)^n \frac{\pi}{6}, \text{ or } n\pi + (-1)^n \frac{3\pi}{2}.
 \end{aligned}$$

$$\begin{aligned}
 27. \quad & \cot \theta - \tan \theta = 2; \\
 & \therefore \cot^2 \theta - 1 = 2 \cot \theta; \\
 & \therefore \cot 2\theta = 1; \\
 & \therefore \theta = \frac{n\pi}{2} + \frac{\pi}{8}.
 \end{aligned}$$

$$\begin{aligned}
 28. \quad & 2 \cos \theta = -1; \therefore \theta = 2n\pi \pm \frac{2\pi}{3} \dots\dots\dots (1), \\
 & 2 \sin \theta = \sqrt{3}; \therefore \theta = n\pi + (-1)^n \frac{2\pi}{3} \dots\dots\dots (2).
 \end{aligned}$$

From (1) we see that the multiple of π must be even, and from (2) that the sign before the second term must be positive when the multiple of π is even;

$$\therefore \theta = 2n\pi + \frac{2\pi}{3}.$$

$$\begin{aligned}
 29. \quad & \sec \theta = \sqrt{2}; \therefore \theta = 2n\pi \pm \frac{\pi}{4} \dots\dots\dots (1), \\
 & \tan \theta = -1; \therefore \theta = n\pi - \frac{\pi}{4} \dots\dots\dots (2).
 \end{aligned}$$

From (1) we see that the multiple of π must be even, and from (2), that the sign before the second term must be negative;

$$\therefore \theta = 2n\pi - \frac{\pi}{4}.$$

EXAMPLES. XIX. b. PAGE 237.

$$\begin{aligned}
 1. \quad & \tan p\theta = \cot q\theta; \\
 & \therefore \tan p\theta = \tan \left(\frac{\pi}{2} - q\theta \right). \\
 & \therefore p\theta = n\pi + \frac{\pi}{2} - q\theta; \\
 & \therefore \theta = \frac{(2n+1)\pi}{2(p+q)}.
 \end{aligned}$$

2.

$$\sin m\theta + \cos n\theta = 0;$$

$$\therefore \cos n\theta = \cos \left(\frac{\pi}{2} + m\theta \right);$$

$$\therefore n\theta = 2\kappa\pi \pm \left(\frac{\pi}{2} + m\theta \right);$$

$$\therefore \theta = \frac{(4\kappa+1)\pi}{2(n-m)}, \text{ or } \frac{(4\kappa-1)\pi}{2(n+m)}.$$

3.

$$\cos \theta - \sqrt{3} \sin \theta = 1;$$

$$\therefore \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta = \frac{1}{2};$$

$$\therefore \cos \left(\theta + \frac{\pi}{3} \right) = \cos \frac{\pi}{3};$$

$$\therefore \theta + \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{3};$$

$$\therefore \theta = 2n\pi, \text{ or } 2n\pi - \frac{2\pi}{3}.$$

4.

$$\sin \theta - \sqrt{3} \cos \theta = 1;$$

$$\therefore \frac{1}{2} \sin \theta - \frac{\sqrt{3}}{2} \cos \theta = \frac{1}{2};$$

$$\therefore \cos \left(\theta + \frac{\pi}{6} \right) = \cos \frac{2\pi}{3};$$

$$\therefore \theta + \frac{\pi}{6} = 2n\pi \pm \frac{2\pi}{3};$$

$$\therefore \theta = 2n\pi + \frac{\pi}{2}, \text{ or } 2n\pi - \frac{5\pi}{6};$$

that is,

$$\theta = 2n\pi + \frac{\pi}{2}, \text{ or } (2n+1)\pi + \frac{\pi}{6}.$$

5.

$$\cos \theta = \sqrt{3} (1 - \sin \theta);$$

$$\therefore \frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta = \frac{\sqrt{3}}{2};$$

$$\therefore \cos \left(\theta - \frac{\pi}{3} \right) = \cos \frac{\pi}{6};$$

$$\therefore \theta - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{6};$$

$$\therefore \theta = 2n\pi + \frac{\pi}{2}, \text{ or } 2n\pi + \frac{\pi}{6}.$$

$$\begin{aligned}
 6. \quad & \sin \theta + \sqrt{3} \cos \theta = \sqrt{2}; \\
 & \therefore \cos \left(\theta - \frac{\pi}{6} \right) = \cos \frac{\pi}{4}; \\
 & \therefore \theta - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}; \\
 & \therefore \theta = 2n\pi + \frac{5\pi}{12}, \text{ or } 2n\pi - \frac{\pi}{12}.
 \end{aligned}$$

$$\begin{aligned}
 7. \quad & \cos \theta - \sin \theta = \frac{1}{\sqrt{2}}; \\
 & \therefore \frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta = \frac{1}{2}, \\
 & \therefore \cos \left(\theta + \frac{\pi}{4} \right) = \cos \frac{\pi}{3}; \\
 & \therefore \theta + \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}; \\
 & \therefore \theta = 2n\pi + \frac{\pi}{12}, \text{ or } 2n\pi - \frac{7\pi}{12}.
 \end{aligned}$$

$$\begin{aligned}
 8. \quad & \cos \theta + \sin \theta + \sqrt{2} = 0; \\
 & \therefore \cos \left(\theta - \frac{\pi}{4} \right) = -1; \\
 & \therefore \theta - \frac{\pi}{4} = 2n\pi \pm \pi; \\
 & \therefore \theta = 2n\pi + \frac{5\pi}{4}, \text{ or } 2n\pi - \frac{3\pi}{4}.
 \end{aligned}$$

$$\begin{aligned}
 9. \quad & \operatorname{cosec} \theta + \cot \theta = \sqrt{3}; \\
 & \therefore 1 + \cos \theta = \sqrt{3} \sin \theta; \\
 & \therefore \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta = -\frac{1}{2}; \\
 & \therefore \cos \left(\theta + \frac{\pi}{3} \right) = -\frac{1}{2}; \\
 & \therefore \theta + \frac{\pi}{3} = 2n\pi \pm \frac{2\pi}{3};
 \end{aligned}$$

$$\text{whence} \quad \theta = 2n\pi + \frac{\pi}{3}, \text{ or } (2n-1)\pi;$$

$$\text{which may be written} \quad \theta = 2n\pi + \frac{\pi}{3}, \text{ or } (2n+1)\pi.$$

$$\begin{aligned}
 10. \quad & \cot \theta - \cot 2\theta = 2; \\
 & \therefore \operatorname{cosec} 2\theta = 2; \\
 & \therefore \theta = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}.
 \end{aligned}$$

11.

$$2 \sin \theta \sin 3\theta = 1;$$

$$\therefore \cos 2\theta - \cos 4\theta = 1;$$

$$\therefore \cos 2\theta = 2 \cos^2 2\theta;$$

$$\therefore \cos 2\theta = 0, \text{ or } \frac{1}{2};$$

$$\therefore \theta = \frac{(2n+1)\pi}{4}, \text{ or } n\pi \pm \frac{\pi}{6}.$$

12.

$$\sin 3\theta = 8 \sin^3 \theta;$$

$$\therefore 3 \sin \theta - 4 \sin^3 \theta = 8 \sin^3 \theta;$$

$$\therefore \sin \theta = 4 \sin^3 \theta;$$

$$\therefore \sin \theta = 0, \text{ or } \pm \frac{1}{2};$$

$$\therefore \theta = n\pi, \text{ or } n\pi \pm \frac{\pi}{6}.$$

13.

$$\tan \theta + \tan 3\theta = 2 \tan 2\theta;$$

$$\therefore \frac{\sin 4\theta}{\cos 3\theta \cos \theta} = \frac{2 \sin 2\theta}{\cos 2\theta};$$

$$\therefore \sin 2\theta = 0; \text{ whence } \theta = \frac{n\pi}{2},$$

or

$$\cos^2 2\theta = \cos \theta \cos 3\theta;$$

$$\therefore 2 \cos^2 2\theta = \cos 4\theta + \cos 2\theta = 2 \cos^2 2\theta - 1 + \cos 2\theta;$$

$$\therefore \cos 2\theta = 1;$$

whence

$$\theta = n\pi;$$

$$\therefore \text{all the values are included in } \theta = \frac{n\pi}{2}.$$

14.

$$\cos \theta - \sin \theta = \cos 2\theta;$$

$$\therefore 2 \sin \frac{3\theta}{2} \sin \frac{\theta}{2} = \sin \theta;$$

$$\therefore \sin \frac{\theta}{2} = 0, \text{ whence } \theta = 2n\pi,$$

or

$$\sin \frac{3\theta}{2} = \cos \frac{\theta}{2};$$

$$\therefore \frac{\theta}{2} = 2n\pi \pm \left(\frac{\pi}{2} - \frac{3\theta}{2} \right);$$

$$\therefore \theta = n\pi + \frac{\pi}{4}, \text{ or } -\theta = 2n\pi - \frac{\pi}{2};$$

$$\therefore \text{the values of } \theta \text{ may be written } 2n\pi, n\pi + \frac{\pi}{4}, 2n\pi + \frac{\pi}{2}.$$

$$\begin{aligned}
 15. \quad & \operatorname{cosec} \theta + \sec \theta = 2\sqrt{2}; \\
 & \therefore \sin \theta + \cos \theta = 2\sqrt{2} \sin \theta \cos \theta; \\
 & \therefore \cos \left(\theta - \frac{\pi}{4} \right) = 2 \sin \theta \cos \theta = \sin 2\theta; \\
 & \therefore \theta - \frac{\pi}{4} = 2n\pi \pm \left(\frac{\pi}{2} - 2\theta \right);
 \end{aligned}$$

whence

$$\theta = \frac{2n\pi}{3} + \frac{\pi}{4}, \text{ or } 2n\pi + \frac{\pi}{4}.$$

$$\begin{aligned}
 16. \quad & \sec \theta - \operatorname{cosec} \theta = 2\sqrt{2}; \\
 & \therefore 1 - \sin 2\theta = 8 \sin^2 \theta \cos^2 \theta = 2 \sin^2 2\theta; \\
 & \therefore \sin 2\theta = -1, \text{ or } \frac{1}{2}; \\
 & \therefore \theta = n\pi - \frac{\pi}{4}, \text{ or } \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}.
 \end{aligned}$$

The equation may also be solved in the same way as Ex. 15.

$$\begin{aligned}
 17. \quad & \frac{1}{\cos 4\theta} - \frac{1}{\cos 2\theta} = 2; \\
 & \cos 2\theta - \cos 4\theta = 2 \cos 4\theta \cos 2\theta \\
 & \qquad \qquad \qquad = \cos 6\theta + \cos 2\theta; \\
 & \therefore \cos 6\theta + \cos 4\theta = 0, \\
 & \qquad 2 \cos 5\theta \cos \theta = 0; \\
 & \therefore \theta = (2n+1) \frac{\pi}{2}, \text{ or } 5\theta = (2n+1) \frac{\pi}{2}.
 \end{aligned}$$

$$\begin{aligned}
 18. \quad & \cos 3\theta + 8 \cos^3 \theta = 0; \\
 & \therefore 4 \cos^3 \theta = \cos \theta; \\
 & \therefore \cos \theta = 0, \text{ or } \pm \frac{1}{2}; \\
 & \therefore \theta = \frac{(2n+1)\pi}{2}, \quad 2n\pi \pm \frac{\pi}{3}, \quad 2n\pi \pm \left(\pi - \frac{\pi}{3} \right);
 \end{aligned}$$

that is, the values of θ are $\frac{(2n+1)\pi}{2}, n\pi \pm \frac{\pi}{3}.$

$$\begin{aligned}
 19. \quad & 1 + \sqrt{3} \tan^2 \theta = (1 + \sqrt{3}) \tan \theta; \\
 & \therefore (\sqrt{3} \tan \theta - 1) (\tan \theta - 1) = 0; \\
 & \therefore \tan \theta = 1, \text{ or } \frac{1}{\sqrt{3}}; \\
 & \therefore \theta = n\pi + \frac{\pi}{4}, \text{ or } n\pi + \frac{\pi}{6}.
 \end{aligned}$$

$$\begin{aligned}
 20. \quad & \tan^3 \theta + \cot^3 \theta = 8 \operatorname{cosec}^3 2\theta + 12; \\
 & \therefore \sin^6 \theta + \cos^6 \theta = 1 + 12 \sin^3 \theta \cos^3 \theta; \\
 & \therefore \sin^4 \theta - \sin^2 \theta \cos^2 \theta + \cos^4 \theta = 1 + 12 \sin^3 \theta \cos^3 \theta; \\
 & \therefore 1 - 3 \sin^2 \theta \cos^2 \theta = 1 + 12 \sin^3 \theta \cos^3 \theta; \\
 & \therefore \sin^2 \theta \cos^2 \theta (4 \sin \theta \cos \theta + 1) = 0;
 \end{aligned}$$

whence $\sin 2\theta = 0$, or $-\frac{1}{2}$;

$$\therefore \theta = \frac{n\pi}{2}, \text{ or } \frac{n\pi}{2} + (-1)^{n+1} \frac{\pi}{12}.$$

$$21. \quad \sin \theta = \sqrt{2} \sin \phi, \quad \sqrt{3} \cos \theta = \sqrt{2} \cos \phi;$$

$$\therefore \text{by squaring and adding we have } \sin^2 \theta + 3 \cos^2 \theta = 2,$$

that is, $1 + 2 \cos^2 \theta = 2$, whence $\cos \theta = \pm \frac{1}{\sqrt{2}}$;

$$\therefore \theta = 2n\pi \pm \frac{\pi}{4}, \text{ or } 2n\pi \pm \left(\pi - \frac{\pi}{4}\right);$$

both of which are included in $\theta = n\pi \pm \frac{\pi}{4}$.

Again, we have $\cos \phi = \frac{\sqrt{3}}{\sqrt{2}} \cos \theta = \pm \frac{\sqrt{3}}{2}$;

$$\therefore \phi = 2n\pi \pm \frac{\pi}{6}, \text{ or } 2n\pi \pm \left(\pi - \frac{\pi}{6}\right),$$

which are both included in $\phi = n\pi \pm \frac{\pi}{6}$.

$$22. \quad \operatorname{cosec} \theta = \sqrt{3} \operatorname{cosec} \phi, \quad \cot \theta = 3 \cot \phi.$$

By squaring and subtracting we have

$$1 = 3 (\operatorname{cosec}^2 \phi - 3 \cot^2 \phi) = 3 (1 - 2 \cot^2 \phi);$$

$$\therefore \cot \phi = \pm \frac{1}{\sqrt{3}}; \text{ whence } \phi = n\pi \pm \frac{\pi}{3}.$$

Also $\cot \theta = \pm \sqrt{3}$; whence $\theta = n\pi \pm \frac{\pi}{6}$.

$$23. \quad \sec \phi = \sqrt{2} \sec \theta, \quad \tan \phi = \sqrt{3} \tan \theta.$$

Subtracting the square of the second equation from the square of the first, we obtain

$$1 = 2 \sec^2 \theta - 3 \tan^2 \theta = 2 - \tan^2 \theta;$$

$$\therefore \tan \theta = \pm 1; \text{ whence } \theta = n\pi \pm \frac{\pi}{4}.$$

Also $\tan \phi = \pm \sqrt{3}$; whence $\phi = n\pi \pm \frac{\pi}{3}$.

24. If
$$\theta + \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{6};$$

then

$$\sin\left(\theta + \frac{\pi}{4}\right) = \sin \frac{\pi}{6};$$

$$\therefore \cos\left(\frac{\pi}{2} - \theta - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right);$$

$$\therefore \cos\left(\theta - \frac{\pi}{4}\right) = \cos \frac{\pi}{3};$$

$$\therefore \theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}.$$

Which proves that the same series of angles are given by the two equations.

25. In the figure on page 252 let OE be drawn bisecting the angle P_1OP_3 , and let the angles P_1OE , P_2OE each be equal to α ;

then the formula $\left(2n + \frac{1}{4}\right)\pi \pm \alpha$ comprises angles whose boundary lines are OP_1 or OP_3 .

Again in the formula $\left(n - \frac{1}{4}\right)\pi + (-1)^n\left(\frac{\pi}{2} - \alpha\right)$ the terms $n\pi + (-1)^n\frac{\pi}{2}$ comprise angles whose boundary lines are OY , whether n be odd or even.

Hence the whole formula comprises angles formed by starting again from OY and turning through an angle $-\frac{\pi}{4} - (-1)^n\alpha$; that is, $-\frac{\pi}{4} \pm \alpha$.

Hence the second formula also comprises angles whose boundary lines are OP_1 or OP_3 .

EXAMPLES. XIX. c. PAGE 242.

1. Let
$$\theta = \sin^{-1} \frac{12}{13}; \text{ then } \operatorname{cosec} \theta = \frac{13}{12}.$$

$$\therefore \cot^2 \theta = \frac{169}{144} - 1 = \frac{25}{144};$$

$$\therefore \theta = \cot^{-1} \frac{5}{12}.$$

2. Let
$$\theta = \operatorname{cosec}^{-1} \frac{17}{8}; \text{ then } \operatorname{cosec} \theta = \frac{17}{8}, \text{ and } \cot \theta = \frac{15}{8}.$$

$$\therefore \theta = \tan^{-1} \frac{8}{15}.$$

3. Let
$$\theta = \tan^{-1} x; \text{ then } \tan \theta = x;$$

$$\therefore \sec^2 \theta = 1 + \tan^2 \theta = 1 + x^2,$$

that is,

$$\sec(\tan^{-1} x) = \sqrt{1 + x^2}.$$

$$4. \quad 2 \tan^{-1} \frac{1}{3} = \tan^{-1} \frac{\frac{2}{3}}{1 - \frac{1}{9}} = \tan^{-1} \frac{3}{4}.$$

$$5. \quad \tan^{-1} \frac{4}{3} - \tan^{-1} 1 = \tan^{-1} \frac{\frac{4}{3} - 1}{1 + \frac{4}{3}} = \tan^{-1} \frac{1}{7}.$$

$$6. \quad \tan^{-1} \frac{2}{11} + \cot^{-1} \frac{24}{7} = \tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} \\ = \tan^{-1} \frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{14}{264}} = \tan^{-1} \frac{1}{2}.$$

$$7. \quad \cot^{-1} \frac{4}{3} - \cot^{-1} \frac{15}{8} = \cot^{-1} \frac{\frac{4}{3} \cdot \frac{15}{8} + 1}{\frac{15}{8} - \frac{4}{3}} = \cot^{-1} \frac{84}{13}.$$

$$8. \quad 2 \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{4} = \tan^{-1} \frac{\frac{2}{5}}{1 - \frac{1}{25}} + \tan^{-1} \frac{1}{4} \\ = \tan^{-1} \frac{5}{12} + \tan^{-1} \frac{1}{4} \\ = \tan^{-1} \frac{\frac{5}{12} + \frac{1}{4}}{1 - \frac{5}{48}} = \tan^{-1} \frac{32}{43}.$$

$$9. \quad \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \tan^{-1} \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = \tan^{-1} 1.$$

Again,
$$\tan^{-1} \frac{5}{6} + \tan^{-1} \frac{1}{11} = \tan^{-1} \frac{\frac{5}{6} + \frac{1}{11}}{1 - \frac{5}{66}} = \tan^{-1} 1.$$

$$\begin{aligned}
 10. \quad \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{18} &= \tan^{-1} \frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{56}} + \tan^{-1} \frac{1}{18} \\
 &= \tan^{-1} \frac{3}{11} + \tan^{-1} \frac{1}{18} = \tan^{-1} \frac{\frac{3}{11} + \frac{1}{18}}{1 - \frac{3}{198}} \\
 &= \tan^{-1} \frac{1}{3} = \cot^{-1} 3.
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \tan^{-1} \frac{3}{5} + \sin^{-1} \frac{3}{5} &= \tan^{-1} \frac{3}{5} + \tan^{-1} \frac{3}{4} \\
 &= \tan^{-1} \frac{\frac{3}{5} + \frac{3}{4}}{1 - \frac{9}{20}} = \tan^{-1} \frac{27}{11}.
 \end{aligned}$$

$$12. \quad 2 \cot^{-1} \frac{5}{4} = 2 \tan^{-1} \frac{4}{5} = \tan^{-1} \frac{\frac{8}{5}}{1 - \frac{16}{25}} = \tan^{-1} \frac{40}{9}.$$

$$\begin{aligned}
 13. \quad 2 \tan^{-1} \frac{8}{15} &= \tan^{-1} \frac{\frac{16}{15}}{1 - \frac{64}{225}} = \tan^{-1} \frac{240}{161} = \sin^{-1} \frac{240}{\sqrt{161^2 + 240^2}} \\
 &= \sin^{-1} \frac{240}{289}.
 \end{aligned}$$

$$14. \quad \text{Let } \sin^{-1} x = \theta; \text{ then } \sin \theta = x, \cos \theta = \sqrt{1 - x^2}.$$

$$\therefore \sin(2 \sin^{-1} x) = \sin 2\theta = 2 \sin \theta \cos \theta = 2x \sqrt{1 - x^2}.$$

$$15. \quad \text{Let } \sin^{-1} \sqrt{\frac{1-x}{2}} = \theta; \text{ then } \sin \theta = \sqrt{\frac{1-x}{2}}.$$

$$\text{Now } \cos 2\theta = 1 - (1 - x) = x; \text{ whence } 2\theta = \cos^{-1} x.$$

$$\text{That is, } \cos^{-1} x = 2 \sin^{-1} \sqrt{\frac{1-x}{2}}.$$

16. Let $\tan^{-1} \sqrt{\frac{x}{a}} = \theta$; then $\tan \theta = \sqrt{\frac{x}{a}}$.

Now $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{a - x}{a + x}$; whence $2\theta = \cos^{-1} \frac{a - x}{a + x}$.

That is, $2 \tan^{-1} \sqrt{\frac{x}{a}} = \cos^{-1} \frac{a - x}{a + x}$.

$$17. \quad 2 \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{7} + 2 \tan^{-1} \frac{1}{5} = 2 \tan^{-1} \frac{\frac{1}{8} + \frac{1}{5}}{1 - \frac{1}{40}} + \tan^{-1} \frac{1}{7}$$

$$= 2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{\frac{2}{3}}{1 - \frac{1}{9}} + \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{28}} = \tan^{-1} 1 = \frac{\pi}{4}.$$

18. Let $\theta = \sin^{-1} a$, $\phi = \cos^{-1} b$;

then $\sin \theta = a$, $\cos \theta = \sqrt{1 - a^2}$, $\cos \phi = b$, $\sin \phi = \sqrt{1 - b^2}$;

$$\therefore \cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi = b \sqrt{1 - a^2} + a \sqrt{1 - b^2};$$

$$\therefore \sin^{-1} a - \cos^{-1} b = \theta - \phi = \cos^{-1} \{b \sqrt{1 - a^2} + a \sqrt{1 - b^2}\}.$$

[In some of the examples which follow diagrams may be used with advantage as in Examples 2 and 3 of Art. 249.]

$$19. \quad \sin^{-1} \frac{4}{5} + \cos^{-1} \frac{2}{\sqrt{5}} = \cot^{-1} \frac{3}{4} + \cot^{-1} 2 = \cot^{-1} \frac{\frac{3}{4} - 1}{\frac{3}{4} + 2} = \cot^{-1} \frac{2}{11}.$$

$$\begin{aligned} 20. \quad \cos^{-1} \frac{63}{65} + 2 \tan^{-1} \frac{1}{5} &= \cos^{-1} \frac{63}{65} + \tan^{-1} \frac{5}{12} \\ &= \tan^{-1} \frac{\sqrt{65^2 - 63^2}}{63} + \tan^{-1} \frac{5}{12} \\ &= \tan^{-1} \frac{16}{63} + \tan^{-1} \frac{5}{12} \\ &= \tan^{-1} \frac{192 + 315}{756 - 80} = \tan^{-1} \frac{507}{676} = \tan^{-1} \frac{3}{4} \\ &= \sin^{-1} \frac{3}{5}. \end{aligned}$$

$$\begin{aligned}
 21. \quad \tan^{-1} m + \tan^{-1} n &= \tan^{-1} \frac{m+n}{1-mn} \\
 &= \cos^{-1} \frac{1-mn}{\sqrt{(m+n)^2 + (1-mn)^2}} = \cos^{-1} \frac{1-mn}{\sqrt{1+m^2+n^2+m^2n^2}} \\
 &= \cos^{-1} \frac{1-mn}{\sqrt{(1+m^2)(1+n^2)}}.
 \end{aligned}$$

$$\begin{aligned}
 22. \quad \text{Let} \quad \cos^{-1} \frac{20}{29} &= \theta, \quad \tan^{-1} \frac{16}{63} = \phi; \\
 \text{then we have} \quad \sin \theta &= \frac{21}{29}, \quad \sin \phi = \frac{16}{65}, \quad \cos \phi = \frac{63}{65}; \\
 \therefore \cos(\theta - \phi) &= \cos \theta \cos \phi + \sin \theta \sin \phi = \frac{20}{29} \cdot \frac{63}{65} + \frac{21}{29} \cdot \frac{16}{65} = \frac{1596}{1885}; \\
 \text{that is,} \quad \cos^{-1} \frac{20}{29} - \tan^{-1} \frac{16}{63} &= \theta - \phi = \cos^{-1} \frac{1596}{1885}.
 \end{aligned}$$

$$23. \quad \text{Let } \cos^{-1} \sqrt{\frac{2}{3}} = \theta, \text{ then } \cos \theta = \sqrt{\frac{2}{3}}, \quad \sin \theta = \frac{1}{\sqrt{3}}.$$

$$\begin{aligned}
 \text{Now} \quad \cos\left(\theta - \frac{\pi}{6}\right) &= \cos \theta \cos \frac{\pi}{6} + \sin \theta \sin \frac{\pi}{6} \\
 &= \sqrt{\frac{2}{3}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}} \cdot \frac{1}{2} \\
 &= \frac{\sqrt{6} + 1}{2\sqrt{3}}.
 \end{aligned}$$

$$\therefore \theta - \frac{\pi}{6} = \cos^{-1} \frac{\sqrt{6} + 1}{2\sqrt{3}};$$

$$\text{that is,} \quad \cos^{-1} \sqrt{\frac{2}{3}} - \cos^{-1} \frac{\sqrt{6} + 1}{2\sqrt{3}} = \frac{\pi}{6}.$$

$$\begin{aligned}
 24. \quad \text{Second side} &= 2 \cdot \frac{x+x^3}{1-x^4} = \frac{2x}{1-x^2} \\
 &= \tan\left(\tan^{-1} \frac{2x}{1-x^2}\right) \\
 &= \tan(2 \tan^{-1} x).
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \text{Second side} &= \tan^{-1} a - \tan^{-1} b + \tan^{-1} b - \tan^{-1} c + \tan^{-1} c \\
 &= \tan^{-1} a.
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \text{Let} \quad \tan^{-1} x &= \alpha, \quad \tan^{-1} y = \beta, \quad \tan^{-1} z = \gamma; \\
 \text{then} \quad \alpha + \beta + \gamma &= \pi;
 \end{aligned}$$

$$\therefore \tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma; \quad [\text{Art. 135, Ex. 2.}]$$

$$\text{that is,} \quad x + y + z = xyz.$$

$$\begin{aligned}
 27. \quad \text{We have } u &= \cot^{-1} \sqrt{\cos \alpha} - \cot^{-1} \sqrt{\frac{1}{\cos \alpha}} \\
 &= \cot^{-1} \frac{1+1}{\sqrt{\frac{1}{\cos \alpha}} - \sqrt{\cos \alpha}} = \cot^{-1} \frac{2\sqrt{\cos \alpha}}{1 - \cos \alpha}; \\
 \therefore \cot^2 u &= \frac{4 \cos \alpha}{(1 - \cos \alpha)^2}; \\
 \therefore \operatorname{cosec}^2 u &= 1 + \cot^2 u = \left(\frac{1 + \cos \alpha}{1 - \cos \alpha} \right)^2; \\
 \therefore \sin u &= \frac{1 - \cos \alpha}{1 + \cos \alpha} = \tan^2 \frac{\alpha}{2}.
 \end{aligned}$$

EXAMPLES. XIX. d. PAGE 244.

$$\begin{aligned}
 1. \quad \sin^{-1} x &= \cos^{-1} x = \sin^{-1} \sqrt{1 - x^2}; \\
 \therefore x &= \sqrt{1 - x^2}; \text{ whence } x = \pm \frac{1}{\sqrt{2}}.
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \tan^{-1} x &= \cot^{-1} x = \tan^{-1} \frac{1}{x}; \\
 \therefore x &= \frac{1}{x}; \text{ whence } x = \pm 1.
 \end{aligned}$$

$$\begin{aligned}
 3. \quad \tan^{-1} (x+1) - \tan^{-1} (x-1) &= \cot^{-1} 2; \\
 \therefore \tan^{-1} \frac{2}{x^2} &= \tan^{-1} \frac{1}{2}; \text{ whence } x = \pm 2.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \cot^{-1} x + \cot^{-1} 2x &= \frac{3\pi}{4}; \\
 \therefore \cot^{-1} \frac{2x^2 - 1}{3x} &= \frac{3\pi}{4}; \\
 \therefore \frac{2x^2 - 1}{3x} &= \cot \frac{3\pi}{4} = -1; \\
 \therefore 2x^2 + 3x - 1 &= 0; \text{ whence } x = \frac{-3 \pm \sqrt{17}}{4}.
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \sin^{-1} x - \cos^{-1} x &= \sin^{-1} (3x - 2); \\
 \therefore \sin (\sin^{-1} x - \cos^{-1} x) &= 3x - 2; \\
 \therefore x^2 - \sqrt{1 - x^2} \cdot \sqrt{1 - x^2} &= 3x - 2;
 \end{aligned}$$

that is,

$$\begin{aligned}
 2x^2 - 1 &= 3x - 2; \\
 \therefore 2x^2 - 3x + 1 &= 0; \text{ whence } x = 1, \text{ or } \frac{1}{2}.
 \end{aligned}$$

$$\begin{aligned}
 6. \quad & \cos^{-1} x - \sin^{-1} x = \cos^{-1} x \sqrt{3}; \\
 & \therefore \cos(\cos^{-1} x - \sin^{-1} x) = x \sqrt{3}; \\
 & \therefore x \sqrt{1-x^2} + x \sqrt{1-x^2} = x \sqrt{3};
 \end{aligned}$$

that is, $2x \sqrt{1-x^2} = x \sqrt{3}$; whence $x=0$, or $\pm \frac{1}{2}$.

$$7. \quad \tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4};$$

$$\therefore \tan^{-1} \frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{x^2-1}{x^2-4}} = \frac{\pi}{4};$$

$$\therefore \frac{2x^2-4}{-3} = 1;$$

$$\therefore 2x^2 = 1, \text{ or } x = \pm \frac{1}{\sqrt{2}}.$$

$$8. \quad 2 \cot^{-1} 2 + \cos^{-1} \frac{3}{5} = \operatorname{cosec}^{-1} x;$$

$$\therefore \cot^{-1} \frac{3}{4} + \cot^{-1} \frac{3}{4} = \operatorname{cosec}^{-1} x;$$

$$\therefore \cot^{-1} \frac{\frac{9}{16} - 1}{\frac{3}{2}} = \operatorname{cosec}^{-1} x;$$

$$\therefore \cot^{-1} -\frac{7}{24} = \operatorname{cosec}^{-1} x = \cot^{-1} \sqrt{x^2-1};$$

$$\therefore \frac{49}{24^2} = x^2 - 1; \text{ whence } x = \pm \frac{25}{24}.$$

$$9. \quad \tan^{-1} x + \tan^{-1} (1-x) = 2 \tan^{-1} \sqrt{x-x^2};$$

$$\therefore \tan^{-1} \frac{1}{1-x+x^2} = \tan^{-1} \frac{2\sqrt{x-x^2}}{1-x+x^2};$$

$$\therefore 1 = 4(x-x^2); \text{ whence } x = \frac{1}{2}.$$

$$10. \quad \cos^{-1} \frac{1-a^2}{1+a^2} - \cos^{-1} \frac{1-b^2}{1+b^2} = 2 \tan^{-1} x;$$

$$\therefore \tan^{-1} \frac{2a}{1-a^2} - \tan^{-1} \frac{2b}{1-b^2} = 2 \tan^{-1} x;$$

$$\therefore 2 \tan^{-1} a - 2 \tan^{-1} b = 2 \tan^{-1} x;$$

$$\therefore \tan^{-1} a - \tan^{-1} b = \tan^{-1} x; \text{ whence } x = \frac{a-b}{1+ab}.$$

$$11. \quad \sin^{-1} \frac{2a}{1+a^2} + \tan^{-1} \frac{2x}{1-x^2} = \cos^{-1} \frac{1-b^2}{1+b^2};$$

$$\therefore \tan^{-1} \frac{2a}{1-a^2} + \tan^{-1} \frac{2x}{1-x^2} = \tan^{-1} \frac{2b}{1-b^2};$$

$$\therefore 2 \tan^{-1} a + 2 \tan^{-1} x = 2 \tan^{-1} b; \text{ whence } x = \frac{b-a}{1+ab}.$$

$$12. \quad \cot^{-1} \frac{x^2-1}{2x} + \tan^{-1} \frac{2x}{x^2-1} + \frac{4\pi}{3} = 0;$$

$$\therefore 2 \tan^{-1} \frac{2x}{x^2-1} + \frac{4\pi}{3} = 0;$$

$$\therefore \tan^{-1} \frac{2x}{1-x^2} = \frac{2\pi}{3};$$

$$\tan^{-1} x = \frac{\pi}{3}; \text{ whence } x = \sqrt{3}.$$

$$13. \quad \sin^{-1} \frac{2ab}{a^2+b^2} = \sin^{-1} \frac{\frac{2b}{a}}{1+\frac{b^2}{a^2}} = \tan^{-1} \frac{\frac{2b}{a}}{1-\frac{b^2}{a^2}};$$

$$\begin{aligned} \therefore \sin^{-1} \frac{2ab}{a^2+b^2} + \sin^{-1} \frac{2cd}{c^2+d^2} &= 2 \tan^{-1} \frac{b}{a} + 2 \tan^{-1} \frac{d}{c} \\ &= 2 \tan^{-1} \frac{\frac{b}{a} + \frac{d}{c}}{1 - \frac{bd}{ac}} \\ &= 2 \tan^{-1} \frac{bc+ad}{ac-bd} \\ &= 2 \tan^{-1} \frac{y}{x} = \sin^{-1} \frac{2xy}{x^2+y^2}, \end{aligned}$$

where

$$y = bc + ad, \quad x = ac - bd.$$

$$\begin{aligned} 14. \quad &\sin [2 \cos^{-1} \{ \cot (2 \tan^{-1} x) \}] = 0; \\ \therefore &\sin \left[2 \cos^{-1} \left\{ \cot \left(\tan^{-1} \frac{2x}{1-x^2} \right) \right\} \right] = 0; \\ \therefore &\sin \left[2 \cos^{-1} \frac{1-x^2}{2x} \right] = 0; \\ \therefore &\frac{1-x^2}{2x} \cdot \sqrt{1 - \left(\frac{1-x^2}{2x} \right)^2} = 0; \end{aligned}$$

whence

$$x = \pm 1, \text{ or } 1-x^2 = \pm 2x;$$

that is,

$$x = \pm 1, \text{ or } \pm (1 \pm \sqrt{2}).$$

$$15. \quad 2 \tan^{-1}(\cos \theta) = \tan^{-1}(2 \operatorname{cosec} \theta);$$

$$\therefore \frac{2 \cos \theta}{1 - \cos^2 \theta} = 2 \operatorname{cosec} \theta;$$

$$\therefore \frac{2 \cos \theta}{\sin^2 \theta} = \frac{2}{\sin \theta};$$

$$\therefore \cot \theta = 1; \text{ whence } \theta = n\pi + \frac{\pi}{4}.$$

$$16. \quad \sin(\pi \cos \theta) = \cos(\pi \sin \theta);$$

$$\therefore \pi \sin \theta = \frac{\pi}{2} \pm \pi \cos \theta;$$

$$\therefore \sin \theta \pm \cos \theta = \frac{1}{2};$$

$$\therefore 1 \pm 2 \sin \theta \cos \theta = \frac{1}{4};$$

$$\therefore \sin 2\theta = \pm \frac{3}{4};$$

$$\therefore 2\theta = \pm \sin^{-1} \frac{3}{4}.$$

$$17. \quad \sin(\pi \cot \theta) = \cos(\pi \tan \theta);$$

$$\therefore \pi \tan \theta = 2n\pi \pm \left(\frac{\pi}{2} - \pi \cot \theta \right);$$

$$\therefore \tan \theta \pm \cot \theta = \frac{4n \pm 1}{2};$$

$$\therefore \frac{\sin^2 \theta \pm \cos^2 \theta}{\sin 2\theta} = \frac{4n \pm 1}{4};$$

that is, either $\cot 2\theta$ or $\operatorname{cosec} 2\theta$ is of the form $\frac{4n+1}{4}$.

$$18. \quad \tan(\pi \cot \theta) = \cot(\pi \tan \theta);$$

$$\therefore \pi \tan \theta = n\pi + \frac{\pi}{2} - \pi \cot \theta;$$

$$\therefore \tan^2 \theta - \frac{2n+1}{2} \tan \theta + 1 = 0;$$

$$\begin{aligned} \therefore \tan \theta &= \frac{2n+1 \pm \sqrt{(2n+1)^2 - 16}}{4} \\ &= \frac{2n+1}{4} \pm \frac{\sqrt{4n^2 + 4n - 15}}{4}. \end{aligned}$$

19.

$$\tan^{-1} x + \cot^{-1} y = \tan^{-1} 3;$$

$$\therefore \frac{x + \frac{1}{y}}{1 - \frac{x}{y}} = 3;$$

$$\therefore y = \frac{3x+1}{3-x};$$

thus we see that the only positive integral values which x may have are 1, 2;

when

$$x=1, \quad y=2;$$

when

$$x=2, \quad y=7;$$

and these are all the positive integral solutions of the equation.

MISCELLANEOUS EXAMPLES. G. PAGE 246.

1. Since

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

we have

$$\frac{a}{4} = \frac{b}{5} = \frac{c}{6};$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{45}{60} = \frac{12}{16}.$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{27}{48} = \frac{9}{16}.$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{5}{40} = \frac{2}{16}.$$

That is, the cosines are in the ratio of 12 : 9 : 2.

2. (1)

$$2 \cos^3 \theta + \sin^2 \theta - 1 = 0;$$

$$\therefore 2 \cos^3 \theta = 1 - \sin^2 \theta = \cos^2 \theta;$$

$$\therefore \cos \theta = 0, \text{ or } \frac{1}{2};$$

$$\therefore \theta = \frac{(2n+1)\pi}{2}, \text{ or } 2n\pi \pm \frac{\pi}{3}.$$

(2)

$$\sec^3 \theta - 2 \tan^2 \theta = 2;$$

$$\therefore \sec^3 \theta = 2(1 + \tan^2 \theta) = 2 \sec^2 \theta;$$

$$\therefore \sec \theta = 2, \text{ since } \sec \theta = 0 \text{ is inadmissible};$$

$$\therefore \theta = 2n\pi \pm \frac{\pi}{3}.$$

3.

$$\tan \beta = 2 \sin \alpha \sin \gamma \operatorname{cosec} (\alpha + \gamma);$$

$$\therefore 2 \cot \beta = \frac{\sin (\alpha + \gamma)}{\sin \alpha \sin \gamma} = \cot \alpha + \cot \gamma;$$

$\therefore \cot \alpha, \cot \beta, \cot \gamma$ are in arithmetical progression.

$$\begin{aligned}
 4. \quad 4r(r_1 + r_2 + r_3) &= \frac{4\Delta^2}{s} \left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \right) \\
 &= 4 \{ (s-b)(s-c) + (s-c)(s-a) + (s-a)(s-b) \} \\
 &= 4 \{ 3s^2 - 2(a+b+c)s + bc + ca + ab \} \\
 &= 4(bc + ca + ab) - 4s^2 \\
 &= 4(bc + ca + ab) - (a+b+c)^2 \\
 &= 2(bc + ca + ab) - (a^2 + b^2 + c^2).
 \end{aligned}$$

$$\begin{aligned}
 5. \quad (1) \quad \tan^{-1} \frac{1}{3} - \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} &= \tan^{-1} \frac{\frac{1}{3} - \frac{1}{5}}{1 + \frac{1}{15}} + \tan^{-1} \frac{1}{7} \\
 &= \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{7} \\
 &= \tan^{-1} \frac{\frac{1}{8} + \frac{1}{7}}{1 - \frac{1}{56}} = \tan^{-1} \frac{3}{11}.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \cos \left(\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{8}{17} \right) &= \frac{4}{5} \cdot \frac{15}{17} - \frac{3}{5} \cdot \frac{8}{17} \\
 &= \frac{36}{85};
 \end{aligned}$$

$$\begin{aligned}
 \therefore \sin^{-1} \frac{3}{5} + \sin^{-1} \frac{8}{17} &= \cos^{-1} \frac{36}{85} \\
 &= \frac{\pi}{2} - \sin^{-1} \frac{36}{85}.
 \end{aligned}$$

$$6. \quad \cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}} = \sqrt{\frac{333 \times 74}{185 \times 222}} = \sqrt{\frac{6}{10}}.$$

$$\begin{aligned}
 \log \cos \frac{C}{2} = \frac{1}{2} (\log 6 - 1) &= \frac{1}{2} (\bar{1} \cdot 7781513) \\
 &= \bar{1} \cdot 8890756;
 \end{aligned}$$

$$\begin{aligned}
 \log \cos 39^\circ 14' &= \bar{1} \cdot 8890644 \\
 \text{diff.} &= 112
 \end{aligned}$$

$$\text{prop}^l. \text{ decrease} = \frac{112}{1032} \times 60'' = 6 \cdot 5''.$$

$$\therefore \frac{C}{2} = 39^\circ 13' 53 \cdot 5'', \text{ and } C = 78^\circ 27' 47''.$$

$$7. \quad \tan(\alpha + \theta) = n \tan(\alpha - \theta);$$

$$\therefore \frac{\tan(\alpha + \theta)}{\tan(\alpha - \theta)} = n;$$

$$\therefore \frac{\tan(\alpha + \theta) - \tan(\alpha - \theta)}{\tan(\alpha + \theta) + \tan(\alpha - \theta)} = \frac{n-1}{n+1};$$

that is,

$$\frac{\sin 2\theta}{\sin 2\alpha} = \frac{n-1}{n+1}.$$

8.

$$8R^2 = a^2 + b^2 + c^2;$$

$$\therefore 2 = \sin^2 A + \sin^2 B + \sin^2 C$$

$$= 2 - \cos(A+B) \cos(A-B) - \cos^2 C$$

$$= 2 + 2 \cos A \cos B \cos C;$$

$$\therefore \cos A \cos B \cos C = 0;$$

that is, one of the angles of the triangle is a right angle.

$$9. \quad \text{Area of inscribed polygon} = nr^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n};$$

$$\text{Area of circumscribed polygon} = nr^2 \tan \frac{\pi}{n};$$

$\therefore$ we have

$$\cos^2 \frac{\pi}{n} = \frac{3}{4};$$

$$\therefore \cos \frac{\pi}{n} = \frac{\sqrt{3}}{2};$$

$$\therefore n = 6.$$

10. Let P, Q be the positions of the boats; then we have

$$\angle PBA = \angle QBC = 45^\circ, \quad \angle PCB = 15^\circ, \quad \angle QCD = 75^\circ.$$

$$\therefore PB = \frac{400 \sin 15^\circ}{\sin 30^\circ} = 200 \sqrt{2} (\sqrt{3} - 1);$$

$$QB = \frac{400 \sin 75^\circ}{\sin 30^\circ} = 200 \sqrt{2} (\sqrt{3} + 1).$$

But PBQ is a right angle;

$$\therefore PQ^2 = PB^2 + QB^2 = 200^2 (12 + 4) = 200^2 \times 4^2;$$

$$\therefore PQ = 800.$$

Again, the distance of P from $AB = PB \sin 45^\circ = 200 (\sqrt{3} - 1)$

$$= 146.4 \text{ yds.},$$

and the distance of Q from $AB = QB \sin 75^\circ = 200 (\sqrt{3} + 1)$

$$= 546.4 \text{ yds.}$$

EXAMPLES. XX. a. PAGE 255.

1. When $\frac{A}{2}$ lies between -135° and -180° , $\sin \frac{A}{2}$ is negative, therefore in the first formula of Art. 254, the negative sign must be taken.

2. $\frac{A}{2}$ lies between 135° and 180° ; and therefore

$\sin \frac{A}{2}$ is positive, $\cos \frac{A}{2}$ is negative.

$$\therefore \sin \frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}} = \sqrt{\frac{1 - \frac{119}{169}}{2}} = \frac{5}{13},$$

$$\cos \frac{A}{2} = -\sqrt{\frac{1 + \cos A}{2}} = -\sqrt{\frac{1 + \frac{119}{169}}{2}} = -\frac{12}{13}.$$

3. Here $\sin \frac{A}{2}$ is negative, and $\cos \frac{A}{2}$ is positive;

$$\therefore \sin \frac{A}{2} = -\sqrt{\frac{1 - \cos A}{2}} = -\sqrt{\frac{1 - \frac{161}{289}}{2}} = -\frac{15}{17},$$

$$\cos \frac{A}{2} = \sqrt{\frac{1 + \cos A}{2}} = \sqrt{\frac{1 + \frac{161}{289}}{2}} = \frac{8}{17}.$$

4. When $\frac{A}{2}$ lies between 135° and 225° , $\cos \frac{A}{2} > \sin \frac{A}{2}$ and is negative;

$$\therefore \sin \frac{A}{2} + \cos \frac{A}{2} = -\sqrt{1 + \sin A},$$

and

$$\sin \frac{A}{2} - \cos \frac{A}{2} = +\sqrt{1 + \sin A}.$$

7. When $\frac{A}{2}$ lies between 45° and 90° , $\sin \frac{A}{2} > \cos \frac{A}{2}$ and is positive;

$$\therefore \sin \frac{A}{2} + \cos \frac{A}{2} = \sqrt{1 + \sin A} = \sqrt{1 + \frac{24}{25}} = \frac{7}{5},$$

and

$$\sin \frac{A}{2} - \cos \frac{A}{2} = \sqrt{1 - \sin A} = \sqrt{1 - \frac{24}{25}} = \frac{1}{5};$$

$$\therefore \sin \frac{A}{2} = \frac{4}{5}, \cos \frac{A}{2} = \frac{3}{5}.$$

8. When $\frac{A}{2}$ lies between 135° and 180° , $\cos \frac{A}{2} > \sin \frac{A}{2}$ and is negative;

$$\therefore \sin \frac{A}{2} + \cos \frac{A}{2} = -\sqrt{1 + \sin A} = -\sqrt{1 - \frac{240}{289}} = -\frac{7}{17},$$

and
$$\sin \frac{A}{2} - \cos \frac{A}{2} = \sqrt{1 - \sin A} = \sqrt{1 + \frac{240}{289}} = \frac{23}{17};$$

$$\therefore \sin \frac{A}{2} = \frac{8}{17}, \cos \frac{A}{2} = -\frac{15}{17}.$$

9. (1) We have

$$\sin A + \cos A = \sqrt{1 + \sin 2A} \dots \dots \dots (i),$$

$$\sin A - \cos A = -\sqrt{1 - \sin 2A} \dots \dots \dots (ii).$$

From (i) we see that of $\sin A$ and $\cos A$ the numerically greater is positive.

From (ii) we see that $\cos A$ is the greater.

Now $\cos A$ is greater than $\sin A$ and positive between the limits $2n\pi - \frac{\pi}{4}$ and $2n\pi + \frac{\pi}{4}$.

(3) We have

$$\sin A + \cos A = -\sqrt{1 + \sin 2A} \dots \dots \dots (i),$$

$$\sin A - \cos A = +\sqrt{1 - \sin 2A} \dots \dots \dots (ii).$$

From (i) we see that of $\sin A$ and $\cos A$ the numerically greater is negative.

From (ii) we see that $\cos A$ is the greater.

Now $\cos A$ is greater than $\sin A$ and negative between the limits $2n\pi + \frac{3\pi}{4}$ and $2n\pi + \frac{5\pi}{4}$.

10. If $\frac{A}{2} = 120^\circ$, $\sin \frac{A}{2} > \cos \frac{A}{2}$ and is positive,

$$\therefore \sin \frac{A}{2} + \cos \frac{A}{2} = \sqrt{1 + \sin A},$$

$$\sin \frac{A}{2} - \cos \frac{A}{2} = \sqrt{1 - \sin A}.$$

$$\therefore 2 \sin \frac{A}{2} = \sqrt{1 + \sin A} + \sqrt{1 - \sin A}.$$

$$\begin{aligned}
 11. \quad \tan 7\frac{1}{2}^\circ &= \frac{1 - \cos 15^\circ}{\sin 15^\circ} = \operatorname{cosec} 15^\circ - \cot 15^\circ \\
 &= \frac{2\sqrt{2}}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} - (2+\sqrt{3}) \\
 &= \sqrt{6} + \sqrt{2} - 2 - \sqrt{3};
 \end{aligned}$$

$$\begin{aligned}
 \cot 142\frac{1}{2}^\circ &= \frac{1 + \cos 285^\circ}{\sin 285^\circ} = \operatorname{cosec} 285^\circ + \cot 285^\circ \\
 &= -\operatorname{cosec} 75^\circ - \cot 75^\circ = -\frac{2\sqrt{2}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} - (2-\sqrt{3}) \\
 &= -\sqrt{2}(\sqrt{3}-1) - 2 + \sqrt{3} = \sqrt{2} + \sqrt{3} - 2 - \sqrt{6}.
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \sin 9^\circ &= \frac{1}{4} \{ \sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}} \} \quad [\text{Art. 260.}] \\
 &= \frac{1}{4} \{ \sqrt{5 \cdot 236080} - \sqrt{2 \cdot 7639320} \} \\
 &= \frac{1}{4} \{ 2 \cdot 288 \dots - 1 \cdot 662 \dots \} \\
 &= \frac{\cdot 626 \dots}{4} = \cdot 156 \dots
 \end{aligned}$$

$$13. (1) \text{ As in Art. 251, } 2 \cos \frac{\pi}{8} = \sqrt{2+\sqrt{2}};$$

$$\text{but} \quad 4 \sin^2 \frac{\pi}{16} = 2 - 2 \cos \frac{\pi}{8} = 2 - \sqrt{2+\sqrt{2}};$$

$$\therefore 2 \sin \frac{\pi}{16} = \sqrt{2 - \sqrt{2+\sqrt{2}}}.$$

$$\begin{aligned}
 (2) \quad \tan 11^\circ 15' &= \frac{1 - \cos 22\frac{1}{2}^\circ}{\sin 22\frac{1}{2}^\circ} = \operatorname{cosec} 22\frac{1}{2}^\circ - \cot 22\frac{1}{2}^\circ \\
 &= \frac{2}{\sqrt{2}-\sqrt{2}} - \frac{1}{\sqrt{2}-1} = \frac{2\sqrt{2+\sqrt{2}}}{\sqrt{2}} - (\sqrt{2}+1) \\
 &= \sqrt{4+2\sqrt{2}} - (\sqrt{2}+1).
 \end{aligned}$$

$$14. (1) \quad \cos \theta + \sin \theta = \sqrt{2} \cos \left(\theta - \frac{\pi}{4} \right).$$

As θ increases from 0 to $\frac{\pi}{4}$, the expression is positive and increases from 1 to $\sqrt{2}$.

As θ increases from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, the expression is positive and decreases from $\sqrt{2}$ to 0.

As θ increases from $\frac{3\pi}{4}$ to $\frac{5\pi}{4}$, the expression is negative and increases numerically from 0 to $-\sqrt{2}$.

As θ increases from $\frac{5\pi}{4}$ to $\frac{7\pi}{4}$, the expression is negative and decreases numerically from $-\sqrt{2}$ to 0.

As θ increases from $\frac{7\pi}{4}$ to 2π , the expression is positive and increases from 0 to 1.

$$(2) \sin \theta - \sqrt{3} \cos \theta = 2 \left(\frac{1}{2} \sin \theta - \frac{\sqrt{3}}{2} \cos \theta \right) = 2 \sin \left(\theta - \frac{\pi}{3} \right).$$

From $\theta=0$ to $\frac{\pi}{3}$, the expression is negative and decreases numerically from $-\sqrt{3}$ to 0.

From $\theta=\frac{\pi}{3}$ to $\frac{\pi}{2} + \frac{\pi}{3}$, or $\frac{5\pi}{6}$, the expression is positive and increases from 0 to 2.

From $\theta=\frac{5\pi}{6}$ to $\frac{\pi}{2} + \frac{5\pi}{6}$, or $\frac{4\pi}{3}$, the expression is positive and decreases from 2 to 0.

From $\theta=\frac{4\pi}{3}$ to $\frac{\pi}{2} + \frac{4\pi}{3}$, or $\frac{11\pi}{6}$, the expression is negative and increases numerically from 0 to -2 .

From $\theta=\frac{11\pi}{6}$ to 2π , the expression is negative and decreases numerically from -2 to $-\sqrt{3}$.

$$15. (1) \text{ The expression } = \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} = -\frac{1}{\cos 2\theta} = -\sec 2\theta \\ = -\sec \phi, \text{ where } \phi = 2\theta.$$

We have therefore to trace the changes in $-\sec \phi$, from $\phi=0$ to 2π .

From 0 to $\frac{\pi}{2}$, the expression is negative and increases numerically from -1 to $-\infty$.

From $\frac{\pi}{2}$ to π , the expression is positive and decreases from ∞ to 1.

From π to $\frac{3\pi}{2}$, the expression is positive and increases from 1 to ∞ .

From $\frac{3\pi}{2}$ to 2π , the expression is negative and decreases numerically from $-\infty$ to -1 .

$$(2) \text{ The expression } = \frac{2 \sin \theta (1 - \cos \theta)}{2 \sin \theta (1 + \cos \theta)} = \tan^2 \frac{\theta}{2}.$$

Now $\frac{\theta}{2}$ varies from 0 to $\frac{\pi}{2}$, so that the expression is positive and increases from 0 to ∞ .

EXAMPLES. XX. b. PAGE 260.

1. Here $\tan \frac{A}{2}$ is negative; hence in the formula $\frac{-1 \pm \sqrt{1 + \tan^2 A}}{\tan A}$, the numerator and denominator must have different signs. But when $A = 320^\circ$, $\tan A$ is negative; therefore we must take the sign which will make the numerator positive,

$$\therefore \tan \frac{A}{2} = \frac{-1 + \sqrt{1 + \tan^2 A}}{\tan A}.$$

3. $\tan A = \pm \sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}}$, and since $\tan A$ is positive the radical must have the positive sign prefixed.

$$\therefore \tan A = \sqrt{\frac{13 - 12}{13 + 12}} = \frac{1}{5}.$$

4. $\cot \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{1 - \cos A}}$; and when $\frac{A}{2}$ lies between 90° and 135° , $\cot \frac{A}{2}$ is negative;

$$\therefore \cot \frac{A}{2} = -\sqrt{\frac{5 - 4}{5 + 4}} = -\frac{1}{3}.$$

5. The general solution of $\cot 2\theta = \cot 2\alpha$ is

$$2\theta = n\pi + 2\alpha; \text{ therefore } \theta = \frac{1}{2}(n\pi + 2\alpha),$$

(1) when $n = 2m$, $\cot \theta = \cot (m\pi + \alpha) = \cot \alpha$,

(2) when $n = 2m + 1$, $\cot \theta = \cot \left(m\pi + \frac{\pi}{2} + \alpha \right) = -\tan \alpha$.

6. $\sin \theta = \sin \alpha$; $\therefore \theta = n\pi + (-1)^n \alpha$; hence in finding $\sin \frac{\theta}{3}$ we have to

consider all the values of $\sin \left(\frac{n\pi}{3} + (-1)^n \frac{\alpha}{3} \right)$. Give to n in succession the values 0, 1, 2, 3, ... Proceeding as in Art. 264 we shall find that the first six angles are

$$\frac{\alpha}{3}, \frac{\pi}{3} - \frac{\alpha}{3}, \frac{2\pi}{3} + \frac{\alpha}{3}, \pi - \frac{\alpha}{3}, \frac{4\pi}{3} + \frac{\alpha}{3}, \frac{5\pi}{3} - \frac{\alpha}{3},$$

and that the other angles are coterminal with one of these.

Now

$$\sin \frac{2\pi}{3} + \frac{\alpha}{3} = \sin \left\{ \pi - \left(\frac{\pi}{3} - \frac{\alpha}{3} \right) \right\} = \sin \frac{\pi - \alpha}{3};$$

$$\sin \left(\pi - \frac{\alpha}{3} \right) = \sin \frac{\alpha}{3};$$

$$\sin \left(\frac{4\pi}{3} + \frac{\alpha}{3} \right) = \sin \left\{ \pi + \left(\frac{\pi}{3} + \frac{\alpha}{3} \right) \right\} = -\sin \frac{\pi + \alpha}{3};$$

$$\sin \left(\frac{5\pi}{3} - \frac{\alpha}{3} \right) = \sin \left\{ 2\pi - \left(\frac{\pi}{3} + \frac{\alpha}{3} \right) \right\} = -\sin \frac{\pi + \alpha}{3}.$$

Thus the values of $\sin \frac{\theta}{3}$ are $\sin \frac{\alpha}{3}$, $\sin \frac{\pi - \alpha}{3}$, $-\sin \frac{\pi + \alpha}{3}$.

7. Here $\theta = n\pi + \alpha$, and we have to find all the values of $\tan \left(\frac{n\pi}{3} + \frac{\alpha}{3} \right)$.

Give to n in succession the values 0, 1, 2, 3, ...; then we shall find that all the angles are coterminal with one of the following:

$$\frac{\alpha}{3}, \quad \frac{\pi}{3} + \frac{\alpha}{3}, \quad \frac{2\pi}{3} + \frac{\alpha}{3}, \quad \pi + \frac{\alpha}{3}, \quad \frac{4\pi}{3} + \frac{\alpha}{3}, \quad \frac{5\pi}{3} + \frac{\alpha}{3},$$

and as in Ex. 6 it may be shewn that the tangents of these angles assume one of the three forms

$$\tan \frac{\alpha}{3}, \quad \tan \frac{\pi + \alpha}{3}, \quad -\tan \frac{\pi - \alpha}{3}.$$

8. Here $3\theta = 2n\pi \pm 3\alpha$, or $\theta = \frac{2n\pi}{3} \pm \alpha$, and we have to find all the values of $\sin \left(\frac{2n\pi}{3} \pm \alpha \right)$.

Now n must be of the form $3m$, or $3m+1$, or $3m-1$.

If $n=3m$, $\sin \left(\frac{2n\pi}{3} \pm \alpha \right) = \sin (2m\pi \pm \alpha) = \pm \sin \alpha;$

if $n=3m+1$, $\sin \left(\frac{2n\pi}{3} \pm \alpha \right) = \sin \left(2m\pi + \frac{2\pi}{3} \pm \alpha \right) = \sin \left(\frac{2\pi}{3} \pm \alpha \right);$

if $n=3m-1$, $\sin \left(\frac{2n\pi}{3} \pm \alpha \right) = \sin \left(2m\pi - \frac{\pi}{3} \pm \alpha \right) = -\sin \left(\frac{\pi}{3} \pm \alpha \right).$

9. Here $3\theta = n\pi + (-1)^n 3\alpha$, or $\theta = \frac{n\pi}{3} + (-1)^n \alpha$, and we have to find all the values of $\cos \left\{ \frac{n\pi}{3} + (-1)^n \alpha \right\}$.

If $n=3m$, $\cos \left\{ \frac{n\pi}{3} + (-1)^n \alpha \right\} = \cos \{ m\pi + (-1)^{3m} \alpha \} = \pm \cos \alpha;$

$$\begin{aligned} \text{if } n=3m+1, \cos \left\{ \frac{n\pi}{3} + (-1)^n \alpha \right\} &= \cos \left\{ m\pi + \frac{\pi}{3} + (-1)^{3m+1} \alpha \right\} \\ &= \pm \cos \left(\frac{\pi}{3} \pm \alpha \right); \end{aligned}$$

$$\begin{aligned} \text{if } n=3m+2, \cos \left\{ \frac{n\pi}{3} + (-1)^n \alpha \right\} &= \cos \left\{ m\pi + \frac{2\pi}{3} + (-1)^{3m+2} \alpha \right\} \\ &= \pm \cos \left(\frac{2\pi}{3} \pm \alpha \right). \end{aligned}$$

EXAMPLES. XXI. a. PAGE 267.

1. Let x =distance in feet, then $x=44 \cot 35' = \frac{44}{\theta}$, nearly, where θ is the radian measure of $35'$.

$$\text{That is,} \quad x = 44 \times \frac{60}{35} \times \frac{180}{\pi} = \frac{44 \times 6 \times 180}{11} \text{ ft.};$$

whence $x=1440$ yds.

$$\begin{aligned} 2. \text{ Here } \quad x &= \frac{22}{3} \cot 24' 30'' = \frac{22}{3} \times \frac{180 \times 60}{77} \text{ ft., nearly} \\ &= 342\frac{2}{7} \text{ yds.} \end{aligned}$$

3. With the figure of Ex. 1, p. 264, we have

$$\begin{aligned} PN &= 840 \tan 1^\circ 30' = 840 \times \frac{3}{2} \times \frac{\pi}{180} \text{ yds., nearly} \\ &= 840 \times \frac{11}{420} = 22 \text{ yds.} \end{aligned}$$

4. Let θ be the required angle, then approximately

$$\begin{aligned} \theta = \tan \theta &= \frac{121}{1760 \times 3 \times 12} = \frac{11}{160 \times 3 \times 12} \text{ radians} \\ &= \frac{11}{160 \times 3 \times 12} \times \frac{180 \times 7}{22} \times 60 \text{ minutes} \\ &= 6' 34''. \end{aligned}$$

5. Let θ be the required angle, then approximately

$$\begin{aligned} \frac{\theta}{2} = \tan \frac{\theta}{2} &= \frac{2}{3000} = \frac{1}{1500} \text{ radians;} \\ \therefore \theta &= \frac{1}{750} \times \frac{180 \times 7}{22} \times 60 \text{ minutes} = 4' 35''. \end{aligned}$$

6. The radian measure of $\frac{1^\circ}{4} = \frac{22}{4 \times 180 \times 7} = \frac{11}{14 \times 180}$;

$$\therefore x = \cdot 625 \cot \frac{1^\circ}{4} = \frac{14 \times 180}{11} \times \cdot 625 \text{ inches, nearly}$$

$$= 11 \text{ ft. } 11 \text{ in.}$$

7. The radian measure of $5' = \frac{5 \times 22}{60 \times 180 \times 7} = \frac{11}{42 \times 180}$;

$$\therefore x = \frac{11}{2} \cot \frac{5'}{2} = \frac{11}{2} \times \frac{84 \times 180}{11} \text{ inches, nearly}$$

$$= 210 \text{ yards.}$$

8. Let θ be the difference between the latitudes, then

$$\theta = \frac{11}{3960} = \frac{1}{360} \text{ radians}$$

$$= \frac{180 \times 7 \times 60}{360 \times 22} \text{ minutes} = 9' 33''.$$

9. See figure of Example 2, page 264.

Let DC be the man, CB the tower, A the point distant 24 feet from the tower;

Let $\angle BAC = \alpha$, $\angle CAD = \theta$;

then $\tan \alpha = \frac{120}{24} = 5,$

$$\tan(\alpha + \theta) = \frac{126}{24} = \frac{21}{4}.$$

But $\tan(\alpha + \theta) = \frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta} = \frac{\tan \alpha + \theta}{1 - \theta \tan \alpha}$, approximately;

$$\therefore \frac{21}{4} = \frac{5 + \theta}{1 - 5\theta};$$

whence $\theta = \frac{1}{109} \text{ radians} = 31 \cdot 5', \text{ nearly.}$

10. See figure of Example 2, page 264.

Let DC be the flagstaff, CB the cliff, and let $DC = x$ feet; then taking the angles as before, we have

$$\tan \alpha = \frac{490}{980} = \frac{1}{2} = \cdot 5, \quad \tan \theta = \cdot 04 \text{ approximately;}$$

$$\therefore \tan(\alpha + \theta) = \frac{\cdot 5 + \cdot 04}{1 - \cdot 02} = \frac{54}{98} = \frac{27}{49};$$

$$\therefore \frac{x + 490}{980} = \frac{27}{49};$$

whence $x = 50;$

that is, the height of the flagstaff is 50 feet.

$$11. (1) n' = \frac{n\pi}{180 \times 60} \text{ radians;}$$

$$\therefore \text{Lt.}_{n=0} \left(\frac{\sin n'}{n} \right) = \frac{n\pi}{180 \times 60} \times \frac{1}{n} = \frac{\pi}{10800}.$$

$$(2) n'' = \frac{n\pi}{180 \times 60 \times 60} \text{ radians;}$$

$$\therefore \text{Lt.}_{n=0} \left(\frac{\sin n''}{n} \right) = \frac{n\pi}{180 \times 60 \times 60} \times \frac{1}{n} = \frac{\pi}{648000}.$$

$$12. \frac{1}{2} nr^2 \sin \frac{2\pi}{n} = \pi r^2 \cdot \frac{n}{2\pi} \cdot \sin \frac{2\pi}{n} = \pi r^2 \left(\sin \frac{2\pi}{n} \div \frac{2\pi}{n} \right);$$

but when $n = \infty$, $\frac{2\pi}{n} = 0$, therefore the limit of $\sin \frac{2\pi}{n} \div \frac{2\pi}{n}$ is unity.

Thus the required limit is πr^2 .

$$13. \text{Lt.}_{\theta=0} \left(\frac{1 - \cos \theta}{\theta \sin \theta} \right) = \text{Lt.}_{\theta=0} \left(\frac{\tan \frac{\theta}{2}}{\theta} \right) = \text{Lt.}_{\theta=0} \left(\frac{1}{2} \cdot \frac{\tan \frac{\theta}{2}}{\frac{\theta}{2}} \right) = \frac{1}{2}.$$

$$14. \text{Lt.}_{\theta=0} \left(\frac{m \sin m\theta - n \sin n\theta}{\tan m\theta + \tan n\theta} \right) = \text{Lt.}_{\theta=0} \left(\frac{m \cdot m\theta - n \cdot n\theta}{m\theta + n\theta} \right) \quad [\text{Art. 268.}]$$

$$= \frac{m^2 - n^2}{m + n} = m - n.$$

$$15. \cos \left(\frac{\pi}{3} + \theta \right) = \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta = \frac{1}{2} - \frac{\sqrt{3}}{200} = .491 \text{ nearly.}$$

$$16. 10' 30'' = \frac{21}{2 \times 60 \times 180} \times \frac{22}{7} = \frac{11}{3600} \text{ radians;}$$

$$\therefore \sin 30^\circ 10' 30'' = \sin \left(\frac{\pi}{6} + \frac{11}{3600} \right)$$

$$= \frac{1}{2} \cos \frac{11}{3600} + \frac{\sqrt{3}}{2} \sin \frac{11}{3600}$$

$$= \frac{1}{2} + \frac{11\sqrt{3}}{7200} = .503 \text{ nearly.}$$

$$17. \cos\left(\frac{\pi}{3} + \theta\right) = .49; \text{ and } \cos\frac{\pi}{3} = .5.$$

$\therefore \theta$ is a very small angle, so that approximately

$$\cos\left(\frac{\pi}{3} + \theta\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\theta;$$

$$\therefore \frac{1}{2} - \frac{\sqrt{3}}{2}\theta = .49;$$

$$\therefore \frac{\sqrt{3}\theta}{2} = .01;$$

$$\therefore \theta = \frac{1}{50\sqrt{3}} = \frac{\sqrt{3}}{150} \text{ radians}$$

$$= \frac{\sqrt{3}}{150} \times \frac{7}{22} \times 180^\circ = \frac{21\sqrt{3}}{55} \text{ degrees}$$

$$= 39.7' \text{ nearly.}$$

EXAMPLES. XXI. b. PAGE 271.

1. Let the distance be x miles; then by the rule on page 269, we have

$$x^2 = \frac{3 \times 96}{2} = 3 \times 48;$$

$$\therefore x = 12;$$

that is, the distance is 12 miles.

2. Let a feet be the height of lighthouse above the sea level;

then $15^2 = \frac{3a}{2}$, or $a = 150$;

that is, the height of lighthouse = 150 feet.

3. Let the distances in miles of the horizon visible from the masts of the ships be x_1, x_2 ;

then $x_1^2 = \frac{3 \times 32\frac{2}{3}}{2} = 49$; $\therefore x_1 = 7$,

$$x_2^2 = \frac{3 \times 42\frac{2}{3}}{2} = 64$$
; $\therefore x_2 = 8$;

$\therefore$ the greatest distance at which one mast can be seen from the other

$$= x_1 + x_2 = 15 \text{ miles.}$$

4. Let the distances in miles of the horizon seen from the two masts be x, y respectively,

then
$$x^2 = \frac{3 \times 54}{2} = 81; \therefore x = 9;$$

also
$$x + y = 20; \text{ whence } y = 11.$$

$$\begin{aligned} \text{Height of mast of second ship} &= \frac{2y^2}{3} \text{ feet} \\ &= \frac{242}{3} \text{ feet} = 80 \text{ ft. } 8 \text{ in.} \end{aligned}$$

5. Let x, y be the distances in miles of the horizon visible from the mast and the lamp respectively;

then
$$x^2 = \frac{3 \times 73\frac{1}{2}}{2}; \therefore x = \frac{21}{2},$$

and
$$x + y = 28; \text{ whence } y = \frac{35}{2}.$$

$$\begin{aligned} \therefore \text{height of lamp} &= \frac{2y^2}{3} = \frac{35^2}{6} = \frac{1225}{6} \text{ ft.} \\ &= 204 \text{ ft. } 2 \text{ in.} \end{aligned}$$

6. From the formula on page 271, we have

$$\text{number of degrees in dip of horizon} = \frac{10}{11} \sqrt{\frac{2 \times 2640}{3 \times 1760}} = \frac{10}{11};$$

$$\therefore \text{dip of the horizon} = 54' 33'', \text{ nearly.}$$

7. The greatest distance at which the light must be visible is the distance of a point exactly opposite a point on the shore midway between two lighthouses, and $3\frac{1}{2}$ miles from it.

$$\text{This distance} = \sqrt{12^2 + \left(\frac{7}{2}\right)^2} = \sqrt{\frac{625}{4}} = \frac{25}{2} \text{ miles.}$$

$$\therefore \text{height of lamp} = \frac{2}{3} \times \frac{625}{4} = \frac{625}{6} \text{ feet} = 104 \text{ ft. } 2 \text{ in.}$$

8. Let the height of the hill be h miles,

then we have
$$1.81 = \frac{10}{11} \sqrt{2h};$$

$$\therefore \frac{20}{11} = \frac{10}{11} \sqrt{2h}; \text{ whence } h = 2;$$

that is, the height of the hill = 2 miles = 10560 feet.

9. Height of hill $= \frac{2 \times (30 \cdot 25)^2}{3}$ feet $\doteq 610$ ft. nearly.

$$\begin{aligned} \text{The dip of the horizon} &= \frac{10}{11} \sqrt{\frac{4 \times 121^2}{3 \times 16} \times \frac{1}{3 \times 1760}} \text{ degrees} \\ &= \frac{5}{12} \sqrt{\frac{11}{10}} \text{ degrees} \\ &= \frac{5}{2} \sqrt{110} \text{ minutes} \\ &= 26' 13'', \text{ nearly.} \end{aligned}$$

10. We have $N = \frac{10}{11} \sqrt{2h}$, where h = height in miles,

$$\begin{aligned} &= \frac{10}{11} \sqrt{\frac{2a}{3 \times 1760}}, \text{ where } a = \text{height in feet,} \\ &= \frac{10}{11} \sqrt{\frac{4x^2}{9 \times 1760}} \\ &= \frac{x}{66} \sqrt{\frac{10}{11}}. \end{aligned}$$

11.
$$\frac{\sin 4\theta \cot \theta}{\text{vers } 2\theta \cot^2 2\theta} = \frac{2 \sin 2\theta \cos 2\theta \cot \theta}{2 \sin^2 \theta \cot^2 2\theta}$$

$$= \frac{\sin^3 2\theta \cos \theta}{\sin^3 \theta \cos 2\theta} = \frac{8 \cos^4 \theta}{\cos 2\theta};$$

$\therefore Lt._{\theta=0} \left(\frac{\sin 4\theta \cot \theta}{\text{vers } 2\theta \cot^2 2\theta} \right) = Lt._{\theta=0} \left(\frac{8 \cos^4 \theta}{\cos 2\theta} \right) = 8.$

12.
$$\frac{1 - \cos \theta + \sin \theta}{1 - \cos \theta - \sin \theta} = \frac{2 \sin^2 \frac{\theta}{2} + \sin \theta}{2 \sin^2 \frac{\theta}{2} - \sin \theta} = \frac{\sin \frac{\theta}{2} + \cos \frac{\theta}{2}}{\sin \frac{\theta}{2} - \cos \frac{\theta}{2}};$$

$\therefore Lt._{\theta=0} \left(\frac{1 - \cos \theta + \sin \theta}{1 - \cos \theta - \sin \theta} \right) = Lt._{\theta=0} \left(\frac{\sin \frac{\theta}{2} + \cos \frac{\theta}{2}}{\sin \frac{\theta}{2} - \cos \frac{\theta}{2}} \right) = -1.$

$$13. \quad (1) \quad Lt_{\theta=a} \left(\frac{\sin \theta - \sin a}{\theta - a} \right) = Lt_{\theta=a} \left(\frac{\sin \frac{\theta-a}{2} \cos \frac{\theta+a}{2}}{\frac{\theta-a}{2}} \right).$$

$$\text{But} \quad Lt_{\theta=a} \left(\frac{\sin \frac{\theta-a}{2}}{\frac{\theta-a}{2}} \right) = 1,$$

[Art. 266.]

$$\therefore \text{required limit} = Lt_{\theta=a} \left(\cos \frac{\theta+a}{2} \right) = \cos a.$$

$$(2) \quad Lt_{\theta=a} \left(\frac{\cos \theta - \cos a}{\theta - a} \right) = Lt_{\theta=a} \left(\frac{\sin \frac{a-\theta}{2} \sin \frac{a+\theta}{2}}{\frac{\theta-a}{2}} \right)$$

$$= Lt_{\theta=a} \left(-\sin \frac{a+\theta}{2} \right) = -\sin a.$$

14. Let $AB=32$, $AC=31$;

$$\text{then} \quad \tan C = \frac{32}{31} = 1 + \frac{1}{31};$$

 $\therefore C$ is a little greater than 45° ;

$$\therefore C = \frac{\pi}{4} + \theta, \text{ where } \theta \text{ is small};$$

$$\therefore \frac{1+\theta}{1-\theta} = \frac{32}{31};$$

$$\therefore \theta = \frac{1}{63} \text{ radians} = \frac{1}{63} \times \frac{7 \times 180}{22} \text{ degrees} = \frac{10^\circ}{11} = 54' 33'';$$

$$\therefore C = 45^\circ 54' 33'', \quad B = 44^\circ 5' 27''.$$

15. We have

$$\angle BPA = a = \angle BAP;$$

$$\therefore AB = BP;$$

$$\therefore \frac{AB}{BC} = \frac{BP}{BC} = \frac{\sin 3a}{\sin a} = 3 - 4 \sin^2 a = 3,$$

since the object is distant and therefore a is small;

that is,

$$AB = 3BC, \text{ nearly.}$$

16. We shall first shew that $\frac{\tan(\theta+h)}{\theta+h} - \frac{\tan \theta}{\theta}$ is positive, h being the radian measure of a small positive angle.

$$\begin{aligned} \text{This fraction} &= \frac{\theta \tan(\theta+h) - \overline{\theta+h} \tan \theta}{\theta(\theta+h)} = \frac{\theta(\tan \overline{\theta+h} - \tan \theta) - h \tan \theta}{\theta(\theta+h)} \\ &= \frac{\frac{\theta \sin h}{\cos \theta \cos(\theta+h)} - \frac{h \sin \theta}{\cos \theta}}{\theta(\theta+h)} = \frac{\theta \sin h - h \sin \theta \cos(\theta+h)}{\theta(\theta+h) \cos \theta \cos(\theta+h)}. \end{aligned}$$

$$\text{Now} \quad \frac{\sin h}{h} > \frac{\sin \theta}{\theta} \quad \text{if } h < \theta \quad [\text{Art. 272.}]$$

and

$$\cos(\theta+h) < 1;$$

$$\therefore \theta \sin h > h \sin \theta \cos(\theta+h);$$

that is, the fraction is positive, and $\therefore \frac{\tan(\theta+h)}{\theta+h} > \frac{\tan \theta}{\theta}$.

$$\therefore \frac{\tan \theta}{\theta} \text{ continually increases as } \theta \text{ increases.}$$

$$\text{When } \theta=0, \frac{\tan \theta}{\theta}=1; \text{ when } \theta=\frac{\pi}{2}, \frac{\tan \theta}{\theta}=\infty.$$

Thus the proposition is established.

MISCELLANEOUS EXAMPLES. H. PAGE 283.

$$1. \quad \cos 2\alpha + \cos 2\beta + 2 \cos(\alpha+\beta) = 2 \cos(\alpha+\beta) \{ \cos(\alpha-\beta) + 1 \}$$

$$= 4 \cos(\alpha+\beta) \cos^2 \frac{\alpha-\beta}{2},$$

$$\sin 2\alpha + \sin 2\beta + 2 \sin(\alpha+\beta) = 2 \sin(\alpha+\beta) \{ \cos(\alpha-\beta) + 1 \}$$

$$= 4 \sin(\alpha+\beta) \cos^2 \frac{\alpha-\beta}{2};$$

$$\therefore \text{hypotenuse} = 4 \cos^2 \frac{\alpha-\beta}{2} \sqrt{\cos^2(\alpha+\beta) + \sin^2(\alpha+\beta)} = 4 \cos^2 \frac{\alpha-\beta}{2}.$$

2. If the in-centre and circum-centre are at equal distances from BC ,

$$\text{we have} \quad R \cos A = r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2};$$

$$\therefore \cos A = \cos A + \cos B + \cos C - 1;$$

$$\therefore \cos B + \cos C = 1.$$

3. Let θ be the required angle; then we have

$$\tan(45^\circ + \theta) = 2;$$

$$\therefore \log \tan(45^\circ + \theta) = \cdot 3010300$$

$$\begin{array}{r} \log \tan 63^\circ 26' = \cdot 3009994 \\ \text{diff.} \qquad \qquad \qquad 306 \end{array}$$

$$\therefore \text{prop}^l. \text{ increase} = \frac{306}{3159} \times 60'' = 5 \cdot 8'' = 6'', \text{ nearly};$$

$$\therefore 45^\circ + \theta = 63^\circ 26' 6'';$$

$$\therefore \text{required angle } 18^\circ 26' 6''.$$

4. Denote each expression by E ; then

$$E^2 = (1 - \sin^2 \alpha)(1 - \sin^2 \beta)(1 - \sin^2 \gamma) = \cos^2 \alpha \cos^2 \beta \cos^2 \gamma;$$

that is,

$$E = \pm \cos \alpha \cos \beta \cos \gamma.$$

5. Let p_1, p_2 be the distances of the chords from the centre, and let r be the radius; then

$$p_1 = r \cos 36^\circ;$$

similarly

$$p_2 = r \cos 72^\circ;$$

$$\therefore \text{distance between the chords} = p_1 - p_2 = r(\cos 36^\circ - \cos 72^\circ)$$

$$= r \left(\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4} \right) = \frac{r}{2}.$$

$$\text{Also the sum of the squares of the chords} = 4r^2 \sin^2 36^\circ + 4r^2 \sin^2 72^\circ$$

$$= 4r^2 \left(1 - \frac{3+\sqrt{5}}{8} + 1 - \frac{3-\sqrt{5}}{8} \right)$$

$$= 5r^2.$$

6. Let A be the point at which the railways meet, then we have to solve a triangle in which $A = 60^\circ$, $a = 43$, $b = 48$. It is easy to see that the solution is ambiguous; hence from the third figure of Art. 148 we have

$$CD = 48 \sin 60^\circ = 24\sqrt{3}, \quad AD = 48 \cos 60^\circ = 24.$$

Also

$$B_2D = \sqrt{43^2 - (24\sqrt{3})^2} = \sqrt{121} = 11.$$

$$\therefore AB_1 = 24 + 11 = 35 \text{ miles,}$$

$$AB_2 = 24 - 11 = 13 \text{ miles.}$$

7.

$$a = \cos^{-1} \frac{x}{a} + \cos^{-1} \frac{y}{b};$$

$$\therefore \cos a = \frac{xy}{ab} - \sqrt{\left(1 - \frac{x^2}{a^2}\right) \left(1 - \frac{y^2}{b^2}\right)};$$

$$\therefore \cos^2 a - \frac{2xy}{ab} \cos a + \frac{x^2 y^2}{a^2 b^2} = \left(1 - \frac{x^2}{a^2}\right) \left(1 - \frac{y^2}{b^2}\right);$$

$$\therefore 1 - \cos^2 a = \frac{x^2}{a^2} - \frac{2xy}{ab} \cos a + \frac{y^2}{b^2};$$

that is,

$$\sin^2 a = \frac{x^2}{a^2} - \frac{2xy}{ab} \cos a + \frac{y^2}{b^2}.$$

8. We have $p = 4R \cos \frac{A}{2}$, $q = 4R \cos \frac{B}{2}$, $r = 4R \cos \frac{C}{2}$;

$$\therefore \frac{a}{p} = \frac{2R \sin A}{4R \cos \frac{A}{2}} = \sin \frac{A}{2};$$

$$\therefore \frac{a^2}{p^2} + \frac{b^2}{q^2} + \frac{c^2}{r^2} + \frac{2abc}{pqr} = \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 1.$$

[Ex. 13, p. 120.]

9. Let P be the top of the tower, and let x be its height;

then

$$PA = \frac{x}{\sin \alpha}, \quad PB = \frac{x}{\sin \beta};$$

and

$$PA^2 + OA^2 = PO^2 = PB^2 + OB^2;$$

$$\therefore \frac{x^2}{\sin^2 \alpha} + a^2 = \frac{x^2}{\sin^2 \beta} + b^2;$$

$$\therefore x^2 = \frac{(a^2 - b^2) \sin^2 \alpha \sin^2 \beta}{\sin^2 \alpha - \sin^2 \beta} = \frac{(a^2 - b^2) \sin^2 \alpha \sin^2 \beta}{\sin (\alpha + \beta) \sin (\alpha - \beta)};$$

hence the height of the tower

$$= \frac{\sqrt{a^2 - b^2} \sin \alpha \sin \beta}{\sqrt{\sin (\alpha + \beta) \sin (\alpha - \beta)}}.$$

10. This example follows readily from the results proved in Examples 18, 24, 25 of XVIII. a.

$$\begin{aligned} r^2 + r_1^2 + r_2^2 + r_3^2 &= (r_1 + r_2 + r_3 - r)^2 + 2r(r_1 + r_2 + r_3) - 2(r_2 r_3 + r_3 r_1 + r_1 r_2) \\ &= 16R^2 + 2\{(r_1 r_2 + r r_3) + (r_2 r_3 + r r_1) + (r_3 r_1 + r r_2)\} \\ &\quad - 4(r_1 r_2 + r_2 r_3 + r_3 r_1) \\ &= 16R^2 + 2(ab + bc + ca) - 4s^2 \\ &= 16R^2 - a^2 - b^2 - c^2. \end{aligned}$$

11. (1) Let $\angle BAD = \theta$; then $\angle CAD = A - \theta$;

$$\therefore \frac{\sin(A - \theta)}{\sin(A + B)} = \frac{CD}{AD} = \frac{BD}{AD} = \frac{\sin \theta}{\sin B};$$

$$\therefore \frac{\sin(A - \theta)}{\sin \theta} = \frac{\sin(A + B)}{\sin B};$$

$$\therefore \cot \theta - \cot A = \cot A + \cot B;$$

that is,

$$\cot BAD = 2 \cot A + \cot B.$$

(2) Draw AM perpendicular to BC , then

$$\begin{aligned} 2 \cot ADC &= \frac{2DM}{AM} = \frac{2DC - 2MC}{AM} \\ &= \frac{(BC - MC) - MC}{AM} = \frac{BM}{AM} - \frac{MC}{AM} \\ &= \cot B - \cot C. \end{aligned}$$

12. See fig. of Art. 223.

Then
$$\frac{a}{p} = \frac{BG + GC}{OG} = \frac{BG}{OG} + \frac{GC}{OG} = \tan C + \tan B;$$

$$\therefore \frac{a}{p} + \frac{b}{q} - \frac{c}{r} = 2 \tan C,$$

and

$$\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 2 (\tan A + \tan B + \tan C);$$

$$\begin{aligned} \therefore 4 \left(\frac{a}{p} + \frac{b}{q} + \frac{c}{r} \right) &= 8 \tan A \tan B \tan C \\ &= \left(\frac{a}{p} + \frac{b}{q} - \frac{c}{r} \right) \left(\frac{b}{q} + \frac{c}{r} - \frac{a}{p} \right) \left(\frac{c}{r} + \frac{a}{p} - \frac{b}{q} \right). \end{aligned}$$

EXAMPLES. XXIII. a. PAGE 291.

1. Here the common difference is $2a$;

$$\therefore S = \frac{\sin na}{\sin a} \sin \frac{a + 2n - 1 a}{2} = \frac{\sin^2 na}{\sin a}.$$

2. Here the common difference is $-\beta$;

$$\therefore S = \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \cos \frac{a + a - n - 1 \beta}{2} = \frac{\sin \frac{n\beta}{2} \cos \left(a - \frac{n-1}{2} \beta \right)}{\sin \frac{\beta}{2}}.$$

3. Here the common difference is $-\frac{\pi}{n}$;

$$\therefore S = \frac{\sin \frac{\pi}{2}}{\sin \frac{\pi}{2n}} \sin \frac{\alpha + \alpha - (n-1)\frac{\pi}{n}}{2} = \frac{-\cos \left(\alpha + \frac{\pi}{2n} \right)}{\sin \frac{\pi}{2n}}.$$

4. Here the common difference is $\frac{\pi}{k}$;

$$\therefore S = \frac{\sin \frac{n\pi}{2k}}{\sin \frac{\pi}{2k}} \cos \frac{\frac{\pi}{k} + \frac{n\pi}{k}}{2} = \frac{\sin \frac{n\pi}{2k} \cos \frac{(n+1)\pi}{2k}}{\sin \frac{\pi}{2k}}.$$

5. The common difference is $\frac{2\pi}{19}$ and the number of terms is 9;

$$\therefore S = \frac{\sin \frac{9\pi}{19}}{\sin \frac{\pi}{19}} \cos \frac{9\pi}{19} = \frac{\sin \frac{18\pi}{19}}{2 \sin \frac{\pi}{19}} = \frac{\sin \left(\pi - \frac{\pi}{19} \right)}{2 \sin \frac{\pi}{19}} = \frac{1}{2}.$$

6. Here the common difference is $\frac{2\pi}{21}$ and the number of terms is 10;

$$\therefore S = \frac{\sin \frac{10\pi}{21}}{\sin \frac{\pi}{21}} \cos \frac{11\pi}{21} = \frac{\sin \frac{22\pi}{21}}{2 \sin \frac{\pi}{21}} = -\frac{1}{2}.$$

7. The common difference is $\frac{\pi}{n}$;

$$\begin{aligned} \therefore S &= \frac{\sin \frac{(n-1)\pi}{2n}}{\sin \frac{\pi}{2n}} \sin \frac{\pi + (n-1)\pi}{2n} \\ &= \frac{\sin \left(\frac{\pi}{2} - \frac{\pi}{2n} \right)}{\sin \frac{\pi}{2n}} \sin \frac{\pi}{2} = \cot \frac{\pi}{2n}. \end{aligned}$$

8. The common difference is $\frac{2\pi}{n}$;

$$\begin{aligned}\therefore S &= \frac{\sin \frac{(2n-1)\pi}{n}}{\sin \frac{\pi}{n}} \cos \frac{\pi + \{2(2n-1) - 1\}\pi}{2n} \\ &= \frac{\sin \left(2\pi - \frac{\pi}{n}\right)}{\sin \frac{\pi}{n}} \cos \frac{(4n-2)\pi}{2n} = -\cos \frac{\pi}{n}.\end{aligned}$$

9. The common difference is $-a$;

$$\therefore S = \frac{\sin na}{\sin \frac{a}{2}} \sin \frac{na + (n-2n-1)a}{2} = \sin na.$$

10. The series may be written

$$\sin \theta + \sin (\pi + 2\theta) + \sin (2\pi + 3\theta) + \sin (3\pi + 4\theta) + \dots$$

$$\begin{aligned}\therefore S &= \frac{\sin \frac{n(\pi + \theta)}{2}}{\sin \frac{\pi + \theta}{2}} \sin \frac{\theta + n-1\pi + n\theta}{2} \\ &= \frac{\sin \frac{n(\pi + \theta)}{2} \sin \left\{ \frac{(n+1)\theta}{2} + \frac{(n-1)\pi}{2} \right\}}{\sin \frac{\pi + \theta}{2}}.\end{aligned}$$

11. The series may be written

$$\cos a + \cos (\pi + a - \beta) + \cos (2\pi + a - 2\beta) + \dots$$

$$\begin{aligned}\therefore S &= \frac{\sin \frac{n(\pi - \beta)}{2}}{\sin \frac{\pi - \beta}{2}} \cos \frac{a + (n-1)(\pi - \beta) + a}{2} \\ &= \frac{\sin \frac{n(\pi - \beta)}{2} \cos \left\{ a + \frac{(n-1)(\pi - \beta)}{2} \right\}}{\sin \frac{\pi - \beta}{2}}.\end{aligned}$$

12. The series may be written

$$\cos a + \cos \left(\frac{\pi}{2} + a - \beta \right) + \cos (\pi + a - 2\beta) + \cos \left(\frac{3\pi}{2} + a - 3\beta \right) + \dots$$

$$\begin{aligned} \therefore S &= \frac{\sin \frac{n(\pi - 2\beta)}{4}}{\sin \frac{\pi - 2\beta}{4}} \cos \frac{2a + (n-1) \left(\frac{\pi}{2} - \beta \right)}{2} \\ &= \frac{\sin \frac{n(\pi - 2\beta)}{4} \cos \left\{ a + \frac{(n-1)(\pi - 2\beta)}{4} \right\}}{\sin \frac{\pi - 2\beta}{4}}. \end{aligned}$$

$$13. \quad S = \frac{1}{2} \{ (\cos \theta - \cos 3\theta) + (\cos \theta - \cos 5\theta) + (\cos \theta - \cos 7\theta) + \dots \}$$

$$= \frac{1}{2} \{ n \cos \theta - (\cos 3\theta + \cos 5\theta + \cos 7\theta + \dots + \cos 2n+1\theta) \}$$

$$= \frac{n \cos \theta}{2} - \frac{\sin n\theta}{2 \sin \theta} \cos (n+2)\theta.$$

$$14. \quad S = \frac{1}{2} \{ (\sin 4a - \sin 2a) + (\sin 8a - \sin 2a) + (\sin 12a - \sin 2a) + \dots \}$$

$$= \frac{1}{2} \{ (\sin 4a + \sin 8a + \sin 12a + \dots + \sin 4na) - n \sin 2a \}$$

$$= \frac{\sin 2na}{2 \sin 2a} \sin 2(n+1)a - \frac{n \sin 2a}{2}.$$

$$15. \quad \sec a \sec 2a = \frac{1}{\cos a \cos 2a} = \operatorname{cosec} a \cdot \frac{\sin (2a - a)}{\cos a \cos 2a}$$

$$= \operatorname{cosec} a (\tan 2a - \tan a).$$

Similarly,

$$\sec 2a \sec 3a = \operatorname{cosec} a (\tan 3a - \tan 2a),$$

.....

$$\sec na \sec (n+1)a = \operatorname{cosec} a \{ \tan (n+1)a - \tan na \}.$$

By addition,

$$S = \operatorname{cosec} a \{ \tan (n+1)a - \tan na \}.$$

$$\begin{aligned}
 16. \quad & \operatorname{cosec} \theta \operatorname{cosec} 3\theta = \operatorname{cosec} 2\theta (\cot \theta - \cot 3\theta), \\
 & \operatorname{cosec} 3\theta \operatorname{cosec} 5\theta = \operatorname{cosec} 2\theta (\cot 3\theta - \cot 5\theta), \\
 & \dots\dots\dots \\
 & \operatorname{cosec} (2n-1)\theta \operatorname{cosec} (2n+1)\theta = \operatorname{cosec} 2\theta \{ \cot (2n-1)\theta - \cot (2n+1)\theta \}.
 \end{aligned}$$

By addition, $S = \operatorname{cosec} 2\theta \{ \cot \theta - \cot (2n+1)\theta \}.$

$$\begin{aligned}
 17. \quad & \tan \frac{a}{2} \sec a = \tan a - \tan \frac{a}{2}, \\
 & \tan \frac{a}{2^2} \sec \frac{a}{2} = \tan \frac{a}{2} - \tan \frac{a}{2^2}, \\
 & \dots\dots\dots \\
 & \tan \frac{a}{2^n} \sec \frac{a}{2^{n-1}} = \tan \frac{a}{2^{n-1}} - \tan \frac{a}{2^n}.
 \end{aligned}$$

By addition, $S = \tan a - \tan \frac{a}{2^n}.$

$$\begin{aligned}
 18. \quad & \cos 2a \operatorname{cosec} 3a = \frac{\cos 2a \sin a}{\sin 3a \sin a} = \frac{1}{2} \cdot \frac{\sin 3a - \sin a}{\sin 3a \sin a} \\
 & = \frac{1}{2} (\operatorname{cosec} a - \operatorname{cosec} 3a), \\
 & \cos 6a \operatorname{cosec} 9a = \frac{1}{2} (\operatorname{cosec} 3a - \operatorname{cosec} 9a), \\
 & \dots\dots\dots \\
 & \cos 3^{n-1} \cdot 2a \operatorname{cosec} 3^n a = \frac{1}{2} (\operatorname{cosec} 3^{n-1} a - \operatorname{cosec} 3^n a).
 \end{aligned}$$

By addition, $S = \frac{1}{2} \operatorname{cosec} a - \operatorname{cosec} 3^n a.$

$$\begin{aligned}
 19. \quad & \sin a \sec 3a = \frac{\sin a}{\cos 3a} = \frac{2 \sin a \cos a}{2 \cos 3a \cos a} \\
 & = \frac{\sin (3a - a)}{2 \cos 3a \cos a} \\
 & = \frac{1}{2} (\tan 3a - \tan a), \\
 & \sin 3a \sec 9a = \frac{1}{2} (\tan 9a - \tan 3a), \\
 & \dots\dots\dots \\
 & \sin 3^{n-1} a \sec 3^n a = \frac{1}{2} (\tan 3^n a - \tan 3^{n-1} a).
 \end{aligned}$$

By addition, $S = \frac{1}{2} (\tan 3^n a - \tan a).$

20. Let AB be the diameter, $P_1, P_2, P_3 \dots P_{n-1}$ the points of division of the arc of the semicircle starting from the end B ; then

$$AP_1 = 2a \cos \frac{\pi}{2n}, \quad AP_2 = 2a \cos \frac{2\pi}{2n}, \dots AP_{n-1} = 2a \cos \frac{(n-1)\pi}{2n};$$

$$\begin{aligned} \therefore \text{sum of distances} &= 2a \left\{ \cos \frac{\pi}{2n} + \cos \frac{2\pi}{2n} + \cos \frac{3\pi}{2n} + \dots + \cos \frac{(n-1)\pi}{2n} \right\} \\ &= \frac{2a \sin \frac{(n-1)\pi}{4n}}{\sin \frac{\pi}{4n}} \cos \frac{\pi}{4} \\ &= \frac{a \left\{ \sin \left(\frac{\pi}{2} - \frac{\pi}{4n} \right) - \sin \frac{\pi}{4n} \right\}}{\sin \frac{\pi}{4n}} \\ &= a \left(\cot \frac{\pi}{4n} - 1 \right). \end{aligned}$$

21. Let $P_1, P_2, P_3 \dots$ be the angular points of the polygon beginning with that nearest to XX' in the quadrant XOY .

Let $p_1, p_2, p_3 \dots$ be perpendiculars from $P_1, P_2, P_3 \dots$ on XX' ,
and $q_1, q_2, q_3 \dots$ be perpendiculars from $P_1, P_2, P_3 \dots$ on YY .

Let $\angle P_1OX = \theta$, and let r be the radius of the circle;

then $p_1 = r \sin \theta$, $p_2 = r \sin \left(\theta + \frac{2\pi}{n} \right)$, $p_3 = r \sin \left(\theta + \frac{4\pi}{n} \right)$, ...;

$$\begin{aligned} \therefore S_p &= r \left\{ \sin \theta + \sin \left(\theta + \frac{2\pi}{n} \right) + \sin \left(\theta + \frac{4\pi}{n} \right) + \dots \text{to } n \text{ terms} \right\} \\ &= r \frac{\sin \frac{\pi}{n}}{\sin \frac{\pi}{n}} \cdot \sin \frac{2\theta + (n-1)\frac{2\pi}{n}}{2} = 0. \end{aligned}$$

Similarly $S_q = 0$.

EXAMPLES. XXIII. b. PAGE 294.

1. $2S = 1 + \cos 2\theta + 1 + \cos 6\theta + 1 + \cos 10\theta + \dots$

$$= n + \frac{\sin 2n\theta}{\sin 2\theta} \cos \frac{2\theta + 2\theta + (n-1)4\theta}{2};$$

$$\therefore S = \frac{n}{2} + \frac{\sin 4n\theta}{4 \sin 2\theta}.$$

$$2. \quad 2S = 1 - \cos 2\alpha + 1 - \cos 2\left(\alpha + \frac{\pi}{n}\right) + 1 - \cos 2\left(\alpha + \frac{2\pi}{n}\right) + \dots$$

$$= n - \frac{\sin \pi}{\sin \frac{\pi}{n}} \cos \frac{4\alpha + (n-1)\frac{2\pi}{n}}{2} = n.$$

$$3. \quad 2S = 1 + \cos 2\alpha + 1 + \cos 2\left(\alpha - \frac{\pi}{n}\right) + 1 + \cos 2\left(\alpha - \frac{2\pi}{n}\right) + \dots$$

$$= n + \frac{\sin \pi}{\sin \frac{\pi}{n}} \cos \frac{4\alpha - (n-1)\frac{2\pi}{n}}{2} = n.$$

$$4. \quad 4S = 3 \sin \theta - \sin 3\theta + 3 \sin 2\theta - \sin 6\theta + 3 \sin 3\theta - \sin 9\theta + \dots$$

$$= 3 \{ \sin \theta + \sin 2\theta + \sin 3\theta + \dots \} - (\sin 3\theta + \sin 6\theta + \sin 9\theta + \dots)$$

$$= \frac{3 \sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} \sin \frac{(n+1)\theta}{2} - \frac{\sin \frac{3n\theta}{2}}{\sin \frac{3\theta}{2}} \sin \frac{3(n+1)\theta}{2};$$

$$\therefore S = \frac{3 \sin \frac{n\theta}{2} \sin \frac{(n+1)\theta}{2}}{4 \sin \frac{\theta}{2}} - \frac{\sin \frac{3n\theta}{2} \sin \frac{3(n+1)\theta}{2}}{\sin \frac{3\theta}{2}}.$$

$$5. \quad 4S = 3 \left\{ \sin \alpha + \sin \left(\alpha + \frac{2\pi}{n} \right) + \sin \left(\alpha + \frac{4\pi}{n} \right) + \dots \right\}$$

$$- \left\{ \sin 3\alpha + \sin 3 \left(\alpha + \frac{2\pi}{n} \right) + \sin 3 \left(\alpha + \frac{4\pi}{n} \right) + \dots \right\}$$

$$= 3 \frac{\sin \pi}{\sin \frac{\pi}{n}} \sin \frac{2\alpha + (n-1)\frac{2\pi}{n}}{2} - \frac{\sin 3\pi}{\sin \frac{3\pi}{n}} \sin \frac{6\alpha + (n-1)\frac{6\pi}{n}}{2}$$

$$= 0.$$

$$\begin{aligned}
 6. \quad 4S &= 3 \left\{ \cos a + \cos \left(a - \frac{2\pi}{n} \right) + \cos \left(a - \frac{4\pi}{n} \right) + \dots \right\} \\
 &\quad + \cos 3a + \cos 3 \left(a - \frac{2\pi}{n} \right) + \cos 3 \left(a - \frac{4\pi}{n} \right) + \dots \\
 &= 3 \frac{\sin \frac{\pi}{n}}{\sin \frac{\pi}{n}} \cos \frac{2a - (n-1) \frac{2\pi}{n}}{2} + \frac{\sin \frac{3\pi}{n}}{\sin \frac{3\pi}{n}} \cos \frac{6a - (n-1) \frac{6\pi}{n}}{2} \\
 &= 0.
 \end{aligned}$$

$$7. \quad \text{We have} \quad \tan \theta = \cot \theta - 2 \cot 2\theta, \quad [\text{XI. d. Ex. 18.}]$$

$$2 \tan 2\theta = 2 \cot 2\theta - 2^2 \cot 2^2 \theta,$$

$$2^2 \tan 2^2 \theta = 2^2 \cot 2^2 \theta - 2^3 \cot 2^3 \theta,$$

$$\dots\dots\dots$$

$$2^{n-1} \tan 2^{n-1} \theta = 2^{n-1} \cot 2^{n-1} \theta - 2^n \cot 2^n \theta;$$

$$\therefore \text{by addition,} \quad S = \cot \theta - 2^n \cot 2^n \theta.$$

$$\begin{aligned}
 8. \quad S &= \frac{1}{2 \cos a \cos 2a} + \frac{1}{2 \cos 2a \cos 3a} + \frac{1}{2 \cos 3a \cos 4a} + \dots \\
 &= \frac{1}{2} \{ \sec a \sec 2a + \sec 2a \sec 3a + \sec 3a \sec 4a + \dots \} \\
 &= \frac{1}{2} \operatorname{cosec} a \{ \tan (n+1) a - \tan a \}. \quad [\text{XXIII. a. Ex. 15.}]
 \end{aligned}$$

$$9. \quad \sin^2 \theta \sin 2\theta = \frac{\sin 2\theta}{2} (1 - \cos 2\theta);$$

$$\therefore \sin^2 \theta \sin 2\theta = \frac{\sin 2\theta}{2} - \frac{\sin 4\theta}{4}.$$

Replacing θ by 2θ , we obtain

$$\frac{1}{2} \sin^2 2\theta \sin 4\theta = \frac{\sin 4\theta}{4} - \frac{\sin 8\theta}{8}.$$

$$\text{Similarly,} \quad \frac{1}{4} \sin^2 4\theta \sin 8\theta = \frac{\sin 8\theta}{8} - \frac{\sin 16\theta}{16};$$

$$\dots\dots\dots$$

$$\frac{1}{2^{n-1}} \sin^2 2^{n-1} \theta \sin 2^n \theta = \frac{\sin 2^n \theta}{2^n} - \frac{\sin 2^{n+1} \theta}{2^{n+1}};$$

$$\therefore \text{by addition,} \quad S = \frac{\sin 2\theta}{2} - \frac{\sin 2^{n+1} \theta}{2^{n+1}}.$$

$$10. \quad 2 \cos \theta \sin^2 \frac{\theta}{2} = \cos \theta (1 - \cos \theta) = 1 - \cos^2 \theta - (1 - \cos \theta);$$

$$\therefore 2 \cos \theta \sin^2 \frac{\theta}{2} = \sin^2 \theta - 2 \sin^2 \frac{\theta}{2}.$$

Replacing θ by $\frac{\theta}{2}$, we obtain

$$2^2 \cos \frac{\theta}{2} \sin^2 \frac{\theta}{2^2} = 2 \sin^2 \frac{\theta}{2} - 2^2 \sin^2 \frac{\theta}{2^2}.$$

$$\text{Similarly,} \quad 2^3 \cos \frac{\theta}{2^2} \sin^2 \frac{\theta}{2^3} = 2^2 \sin^2 \frac{\theta}{2^2} - 2^3 \sin^2 \frac{\theta}{2^3};$$

.....

$$2^n \cos \frac{\theta}{2^{n-1}} \sin^2 \frac{\theta}{2^n} = 2^{n-1} \sin^2 \frac{\theta}{2^{n-1}} - 2^n \sin^2 \frac{\theta}{2^n};$$

$$\therefore \text{ by addition,} \quad S = \sin^2 \theta - 2^n \sin^2 \frac{\theta}{2^n}.$$

11. We have

$$\tan^{-1} \frac{x}{n(n+1)+x^2} = \tan^{-1} \frac{\frac{x}{n} - \frac{x}{n+1}}{1 + \frac{x^2}{n(n+1)}} = \tan^{-1} \frac{x}{n} - \tan^{-1} \frac{x}{n+1},$$

and hence

$$\tan^{-1} \frac{x}{1 \cdot 2 + x^2} = \tan^{-1} x - \tan^{-1} \frac{x}{2};$$

$$\tan^{-1} \frac{x}{2 \cdot 3 + x^2} = \tan^{-1} \frac{x}{2} - \tan^{-1} \frac{x}{3};$$

.....

$$\tan^{-1} \frac{x}{n(n+1)+x^2} = \tan^{-1} \frac{x}{n} - \tan^{-1} \frac{x}{n+1};$$

$$\therefore \text{ by addition,} \quad S = \tan^{-1} x - \tan^{-1} \frac{x}{n+1}.$$

12.

$$\begin{aligned} \tan^{-1} \frac{1}{1+n+n^2} &= \tan^{-1} \frac{(n+1)-n}{1+n(n+1)} \\ &= \tan^{-1} (n+1) - \tan^{-1} n. \end{aligned}$$

$$\therefore \tan^{-1} \frac{1}{1+1+1^2} = \tan^{-1} 2 - \tan^{-1} 1;$$

$$\tan^{-1} \frac{1}{1+2+2^2} = \tan^{-1} 3 - \tan^{-1} 2;$$

.....

$$\tan^{-1} \frac{1}{1+n+n^2} = \tan^{-1} (n+1) - \tan^{-1} n;$$

$$\therefore S = \tan^{-1} (n+1) - \tan^{-1} 1 = \tan^{-1} (n+1) - \frac{\pi}{4}.$$

$$13. \quad \tan^{-1} \frac{2n}{2+n^2+n^4} = \tan^{-1} \frac{(1+n+n^2) - (1-n+n^2)}{1 + (1+n+n^2)(1-n+n^2)} \\ = \tan^{-1} (1+n+n^2) - \tan^{-1} (1-n+n^2).$$

$$\therefore \tan^{-1} \frac{2}{2+1^2+1^4} = \tan^{-1} 3 - \tan^{-1} 1;$$

$$\tan^{-1} \frac{4}{2+2^2+2^4} = \tan^{-1} 6 - \tan^{-1} 3;$$

.....

$$\tan^{-1} \frac{2n}{2+n^2+n^4} = \tan^{-1} (1+n+n^2) - \tan^{-1} (1-n+n^2);$$

$$\therefore S = \tan^{-1} (1+n+n^2) - \tan^{-1} 1$$

$$= \tan^{-1} (1+n+n^2) - \frac{\pi}{4}.$$

$$14. \quad \tan^{-1} \frac{2n}{1-n^2+n^4} = \tan^{-1} \frac{n^2+n - (n^2-n)}{1 + (n^2+n)(n^2-n)} \\ = \tan^{-1} (n^2+n) - \tan^{-1} (n^2-n).$$

$$\therefore \tan^{-1} \frac{2}{1-1^2+1^4} = \tan^{-1} 2 - \tan^{-1} 0;$$

$$\tan^{-1} \frac{4}{1-2^2+2^4} = \tan^{-1} 6 - \tan^{-1} 2;$$

.....

$$\tan^{-1} \frac{2n}{1-n^2+n^4} = \tan^{-1} (n^2+n) - \tan^{-1} (n^2-n);$$

$$\therefore S = \tan^{-1} (n^2+n).$$

15. Let O be the point on the circumference of the circle, and $P, Q, R, \dots$ the vertices of the polygon beginning with that nearest to O . Let L be the other extremity of the diameter through O , and let $\angle OLP = \theta$;

then $OP = 2r \sin \theta$, $OQ = 2r \sin \left(\theta + \frac{\pi}{n} \right)$, $OR = 2r \sin \left(\theta + \frac{2\pi}{n} \right)$, ...

$\therefore$ sum of the squares of the chords

$$= 4r^2 \left\{ \sin^2 \theta + \sin^2 \left(\theta + \frac{\pi}{n} \right) + \sin^2 \left(\theta + \frac{2\pi}{n} \right) + \dots \text{ to } n \text{ terms} \right\}$$

$$= 2r^2 \left\{ n - \cos 2\theta - \cos 2 \left(\theta + \frac{\pi}{n} \right) - \cos 2 \left(\theta + \frac{2\pi}{n} \right) - \dots \right\}$$

$$= 2nr^2 - 2r^2 \frac{\sin \pi}{\sin \frac{\pi}{n}} \cos \left(2\theta + (n-1) \frac{\pi}{n} \right)$$

$$= 2nr^2.$$

16. Let O be the centre of the inscribed circle, and let $\angle PON_1 = \theta$, where ON_1 is the perpendicular from O parallel to PA_1 ; let $OP = x$;

$$\text{then } PA_1 = r - x \cos \theta, \quad PA_2 = r - x \cos \left(\theta + \frac{\pi}{n} \right),$$

$$PA_3 = r - x \cos \left(\theta + \frac{2\pi}{n} \right), \quad PA_4 = r - x \cos \left(\theta + \frac{3\pi}{n} \right), \dots$$

$$\therefore PA_1 + PA_3 + \dots + PA_{2n-1}$$

$$= nr - x \left\{ \cos \theta + \cos \left(\theta + \frac{2\pi}{n} \right) + \cos \left(\theta + \frac{4\pi}{n} \right) + \dots \text{ to } n \text{ terms} \right\}$$

$$= nr.$$

[Art. 297.]

$$\text{Similarly } PA_2 + PA_4 + \dots + PA_{2n}$$

$$= nr - x \left\{ \cos \left(\theta + \frac{\pi}{n} \right) + \cos \left(\theta + \frac{3\pi}{n} \right) + \dots \text{ to } n \text{ terms} \right\}$$

$$= nr.$$

17. Let Q be the other extremity of the diameter through P , and let $\angle PQA_1 = \theta$, and let r be the radius of the circle; then

$$PA_1 = 2r \sin \theta, \quad PA_2 = 2r \sin \left(\theta + \frac{\pi}{2n+1} \right), \quad PA_3 = 2r \sin \left(\theta + \frac{2\pi}{2n+1} \right), \dots$$

$$\therefore PA_1 + PA_3 + \dots + PA_{2n+1}$$

$$= 2r \left\{ \sin \theta + \sin \left(\theta + \frac{2\pi}{2n+1} \right) + \sin \left(\theta + \frac{4\pi}{2n+1} \right) + \dots \text{ to } n+1 \text{ terms} \right\}$$

$$= 2r \frac{\sin \frac{(n+1)\pi}{2n+1}}{\sin \frac{\pi}{2n+1}} \sin \left(\theta + \frac{n\pi}{2n+1} \right);$$

$$\text{and } PA_2 + PA_4 + \dots + PA_{2n}$$

$$= 2r \left\{ \sin \left(\theta + \frac{\pi}{2n+1} \right) + \sin \left(\theta + \frac{3\pi}{2n+1} \right) + \dots \text{ to } n \text{ terms} \right\}$$

$$= 2r \frac{\sin \frac{n\pi}{2n+1}}{\sin \frac{\pi}{2n+1}} \sin \frac{1}{2} \left(2\theta + \frac{\pi}{2n+1} + \frac{(2n-1)\pi}{2n+1} \right)$$

$$= 2r \frac{\sin \frac{(n+1)\pi}{2n+1}}{\sin \frac{\pi}{2n+1}} \sin \left(\theta + \frac{n\pi}{2n+1} \right).$$

$$\therefore PA_1 + PA_3 + \dots + PA_{2n+1} = PA_2 + PA_4 + \dots + PA_{2n}.$$

18. Let $p_1, p_2, p_3, \dots, p_n$ be the perpendiculars; then as in Ex. 16, we have

$$p_1 = r - r \cos \theta, \quad p_2 = r - r \cos \left(\theta + \frac{2\pi}{n} \right), \quad p_3 = r - r \cos \left(\theta + \frac{4\pi}{n} \right), \quad \dots$$

$$\begin{aligned} \text{(i)} \quad \Sigma p^2 &= \{r - r \cos \theta\}^2 + \left\{r - r \cos \left(\theta + \frac{2\pi}{n} \right)\right\}^2 + \dots \text{ to } n \text{ terms} \\ &= nr^2 - 2r^2 \left\{ \cos \theta + \cos \left(\theta + \frac{2\pi}{n} \right) + \dots \right\} \\ &\quad + r^2 \left\{ \cos^2 \theta + \cos^2 \left(\theta + \frac{2\pi}{n} \right) + \dots \right\}. \end{aligned}$$

$$\text{Now} \quad \cos \theta + \cos \left(\theta + \frac{2\pi}{n} \right) + \dots = 0; \quad [\text{Art. 297.}]$$

$$\begin{aligned} \therefore \Sigma p^2 &= nr^2 + \frac{r^2}{2} \left\{ n + \cos 2\theta + \cos 2 \left(\theta + \frac{2\pi}{n} \right) + \dots \right\} \\ &= nr^2 + \frac{nr^2}{2} = \frac{3nr^2}{2}. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \Sigma p^3 &= nr^3 - 3r^3 \left\{ \cos \theta + \cos \left(\theta + \frac{2\pi}{n} \right) + \dots \text{ to } n \text{ terms} \right\} \\ &\quad + \frac{3r^3}{2} \left\{ n + \cos 2\theta + \cos 2 \left(\theta + \frac{2\pi}{n} \right) + \dots \right\} \\ &\quad - \frac{r^3}{4} \left\{ 3 \cos \theta + 3 \cos \left(\theta + \frac{2\pi}{n} \right) + \dots \right. \\ &\quad \left. + \cos 3\theta + \cos 3 \left(\theta + \frac{2\pi}{n} \right) + \dots \right\} \\ &= nr^3 + \frac{3nr^3}{2} = \frac{5nr^3}{2}, \end{aligned}$$

since all the series of cosines vanish by Art. 297.

EXAMPLES. XXIV. a. PAGE 301.

1. We have

$$\frac{1}{a} \cos \alpha + \frac{1}{b} \sin \alpha = \frac{1}{c}, \quad \frac{1}{a} \cos \beta + \frac{1}{b} \sin \beta = \frac{1}{c};$$

and the required result follows at once by cross multiplication as in Ex. 1, p. 297.

$$4. \quad \cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$= \frac{c^2 - b^2}{a^2 + b^2} - \frac{c^2 - a^2}{a^2 + b^2},$$

as in Example 2, page 297.

$$\begin{aligned}
 5. \quad \cos^2 \frac{a-\beta}{2} &= \frac{1+\cos(a-\beta)}{2} = \frac{1+\cos a \cos \beta + \sin a \sin \beta}{2} \\
 &= \frac{(a^2+b^2) + (c^2-b^2) + (c^2-a^2)}{2(a^2+b^2)} \\
 &= \frac{c^2}{a^2+b^2}.
 \end{aligned}$$

$$6. \quad \sin 2a + \sin 2\beta = 2 \sin(a+\beta) \cos(a-\beta).$$

From Example 4, we find that $\sin(a+\beta) = \frac{2ab}{a^2+b^2}$;

and
$$\cos(a-\beta) = \frac{c^2-b^2}{a^2+b^2} + \frac{c^2-a^2}{a^2+b^2} = \frac{2c^2-a^2-b^2}{a^2+b^2}.$$

$$\begin{aligned}
 7. \quad \sin^2 a + \sin^2 \beta &= (\sin a + \sin \beta)^2 - 2 \sin a \sin \beta \\
 &= \left(\frac{2bc}{a^2+b^2} \right)^2 - \frac{2(c^2-a^2)}{a^2+b^2}. \quad [\text{See p. 298.}]
 \end{aligned}$$

8. Here a and β are solutions of $a \cos \theta + b \sin \theta = c$; hence as in Example 4, we find

$$\cos(a+\beta) = \frac{a^2-b^2}{a^2+b^2}, \text{ and therefore } \sin(a+\beta) = \frac{2ab}{a^2+b^2}.$$

$$\begin{aligned}
 \text{Again,} \quad \cot a + \cot \beta &= \frac{\sin(a+\beta)}{\sin a \sin \beta} \\
 &= \frac{2ab}{a^2+b^2} \div \frac{c^2-a^2}{a^2+b^2}. \quad [\text{See p. 298.}]
 \end{aligned}$$

9. By squaring and adding, we have

$$2 + 2 \cos(\theta - \phi) = a^2 + b^2.$$

$$\text{Again,} \quad \cos(\theta + \phi) = \frac{a^2 - b^2}{a^2 + b^2}. \quad [\text{Ex. 4, p. 299.}]$$

$$\text{And} \quad 2 \cos \theta \cos \phi = \cos(\theta + \phi) + \cos(\theta - \phi);$$

$$\begin{aligned}
 \therefore 4 \cos \theta \cos \phi &= \frac{2(a^2 - b^2)}{a^2 + b^2} + a^2 + b^2 - 2 \\
 &= \frac{(a^2 + b^2)^2 - 4b^2}{a^2 + b^2}.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \cos 2\theta + \cos 2\phi &= 2 \cos(\theta + \phi) \cos(\theta - \phi) \\
 &= \frac{a^2 - b^2}{a^2 + b^2} (a^2 + b^2 - 2),
 \end{aligned}$$

as in the preceding example.

$$\begin{aligned}
 11. \quad \tan \theta + \tan \phi &= \frac{\sin (\theta + \phi)}{\cos \theta \cos \phi} \\
 &= \frac{2ab}{a^2 + b^2} \div \frac{(a^2 + b^2)^2 - 4b^2}{4(a^2 + b^2)}.
 \end{aligned}$$

[See Ex. 4, p. 299 and Ex. 9 above.]

$$12. \quad \tan \frac{\theta}{2} + \tan \frac{\phi}{2} = \frac{\sin \frac{\theta + \phi}{2}}{\cos \frac{\theta}{2} \cos \frac{\phi}{2}} = \frac{2 \sin \frac{\theta + \phi}{2}}{\cos \frac{\theta + \phi}{2} + \cos \frac{\theta - \phi}{2}}.$$

On multiplying numerator and denominator by $2 \cos \frac{\theta + \phi}{2}$, this last fraction becomes

$$\frac{2 \sin (\theta + \phi)}{1 + \cos (\theta + \phi) + \cos \theta + \cos \phi}.$$

By substituting for $\sin (\theta + \phi)$ and $\cos (\theta + \phi)$ the values found in Ex. 4, p. 299, we obtain

$$\frac{4ab}{a^2 + b^2} \div \left(1 + \frac{a^2 - b^2}{a^2 + b^2} + a \right),$$

which reduces to

$$\frac{4b}{a^2 + b^2 + 2a}.$$

$$\begin{aligned}
 13. \quad \text{The expression} &= \sin^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma \\
 &= -\cos (\alpha + \beta) \cos (\alpha - \beta) - \cos^2 \gamma + \{ \cos (\alpha + \beta) + \cos (\alpha - \beta) \} \cos \gamma \\
 &= -\{ \cos (\alpha + \beta) - \cos \gamma \} \{ \cos (\alpha - \beta) - \cos \gamma \} \\
 &= 4 \sin \frac{\alpha + \beta + \gamma}{2} \sin \frac{\alpha + \beta - \gamma}{2} \sin \frac{\beta + \gamma - \alpha}{2} \sin \frac{\gamma + \alpha - \beta}{2};
 \end{aligned}$$

from which the second part of the question easily follows.

$$\begin{aligned}
 14. \quad \text{The expression} &= \sin^2 \alpha + (\sin^2 \beta - \sin^2 \gamma) + 2 \sin \alpha \sin \beta \cos \gamma \\
 &= \sin^2 \alpha + \sin (\beta + \gamma) \sin (\beta - \gamma) + \sin \alpha \{ \sin (\beta + \gamma) + \sin (\beta - \gamma) \} \\
 &= \{ \sin \alpha + \sin (\beta + \gamma) \} \{ \sin \alpha + \sin (\beta - \gamma) \} \\
 &= 4 \sin \frac{\alpha + \beta + \gamma}{2} \cos \frac{\beta + \gamma - \alpha}{2} \sin \frac{\alpha + \beta - \gamma}{2} \cos \frac{\alpha - \beta + \gamma}{2}.
 \end{aligned}$$

$$\begin{aligned}
 15. \quad & \text{The expression} = \sin^2 a - (\cos^2 \beta - \sin^2 \gamma) - 2 \sin a \sin \beta \sin \gamma \\
 & = \sin^2 a - \cos (\beta + \gamma) \cos (\beta - \gamma) - \sin a \{ \cos (\beta - \gamma) - \cos (\beta + \gamma) \} \\
 & = \{ \sin a + \cos (\beta + \gamma) \} \{ \sin a - \cos (\beta - \gamma) \} \\
 & = - \left\{ \cos \left(\frac{\pi}{2} - a \right) + \cos (\beta + \gamma) \right\} \left\{ \cos \left(\frac{\pi}{2} + a \right) + \cos (\beta - \gamma) \right\} \\
 & = - 4 \cos \left(\frac{\beta + \gamma - a}{2} + \frac{\pi}{4} \right) \cos \left(\frac{\beta + \gamma + a}{2} - \frac{\pi}{4} \right) \cos \left(\frac{a + \beta - \gamma}{2} + \frac{\pi}{4} \right)
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \tan^2 \frac{\theta}{2} &= \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{1 - \cos a + \cos \beta - \cos a \cos \beta}{1 + \cos a - \cos \beta - \cos a \cos \beta} \\
 &= \frac{(1 - \cos a)(1 + \cos \beta)}{(1 + \cos a)(1 - \cos \beta)} = \tan^2 \frac{a}{2} \cot^2 \frac{\beta}{2}.
 \end{aligned}$$

$$17. \quad \tan^2 \theta = \frac{2 \sin a \sin \beta}{1 + \cos (a + \beta)} = \frac{2 \sin a \sin \beta}{1 + \cos a \cos \beta - \sin a \sin \beta};$$

$$\therefore 1 + \tan^2 \theta = \frac{1 + \cos a \cos \beta + \sin a \sin \beta}{1 + \cos a \cos \beta - \sin a \sin \beta};$$

that is,
$$\sec^2 \theta = \frac{1 + \cos (a - \beta)}{1 + \cos (a + \beta)} = \frac{\cos^2 \frac{a - \beta}{2}}{\cos^2 \frac{a + \beta}{2}}.$$

Taking the positive value of the square root, we have

$$\cos \theta = \frac{\cos \frac{a + \beta}{2}}{\cos \frac{a - \beta}{2}}.$$

$$\therefore \tan^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{\sin \frac{a}{2} \sin \frac{\beta}{2}}{\cos \frac{a}{2} \cos \frac{\beta}{2}}.$$

18. Here $\tan \theta = \frac{\sin a \cos \beta}{\cos a + \sin \beta};$

$$\therefore \sec^2 \theta = 1 + \frac{\sin^2 a \cos^2 \beta}{(\cos a + \sin \beta)^2} = \frac{(\cos a + \sin \beta)^2 + (1 - \cos^2 a)(1 - \sin^2 \beta)}{(\cos a + \sin \beta)^2}$$

$$= \frac{1 + 2 \cos a \sin \beta + \cos^2 a \sin^2 \beta}{(\cos a + \sin \beta)^2}.$$

Taking the positive value of the square root, we have

$$\cos \theta = \frac{\cos a + \sin \beta}{1 + \cos a \sin \beta}.$$

$$\therefore \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{(1 - \cos a)(1 - \sin \beta)}{(1 + \cos a)(1 + \sin \beta)}$$

$$= \frac{(1 - \cos a) \left\{ 1 - \cos \left(\frac{\pi}{2} - \beta \right) \right\}}{(1 + \cos a) \left\{ 1 + \cos \left(\frac{\pi}{2} - \beta \right) \right\}};$$

$$\therefore \tan^2 \frac{\theta}{2} = \tan^2 \frac{a}{2} \tan^2 \left(\frac{\pi}{4} - \frac{\beta}{2} \right).$$

19. This identity will be established if we shew that

$$\Sigma \sin^3 A \sin B \sin C = 2 \sin A \sin B \sin C (1 + \cos A \cos B \cos C),$$

that is, if $\Sigma \sin^2 A = 2 (1 + \cos A \cos B \cos C).$

$$\text{Now } \sin^2 A + \sin^2 B + \sin^2 C = \frac{1}{2} (3 - \cos 2A - \cos 2B - \cos 2C)$$

$$= \frac{1}{2} (4 + 4 \cos A \cos B \cos C).$$

[See XII. d. Ex. 9.]

20. Expressing a, b, c in terms of R , this identity will be proved if we shew that

$$\Sigma \sin A \cos^3 A = 4 \sin A (1 - 4 \cos A \cos B \cos C).$$

$$\text{Now } 8 \Sigma \sin A \cos^3 A = 4 \Sigma \cos^2 A \sin 2A$$

$$= 2 \Sigma (1 + \cos 2A) \sin 2A$$

$$= 2 \Sigma \sin 2A + \Sigma \sin 4A.$$

$$\text{Now } \Sigma \sin 2A = 4 \Pi \sin A,$$

and

$$\Sigma \sin 4A = -4 \Pi \sin 2A; \quad [\text{Ex. 7, p. 301}];$$

$$\therefore \Sigma \sin A \cos^3 A = \Pi \sin A - 4 \Pi \sin A \cos A$$

$$= \Pi \sin A (1 - 4 \cos A \cos B \cos C).$$

$$\begin{aligned}
 21. \quad \Sigma a^3 \cos (B-C) &= 2R \Sigma a^2 \sin A \cos (B-C) \\
 &= 2R \Sigma a^2 \sin (B+C) \cos (B-C) \\
 &= R \Sigma a^2 (\sin 2B + \sin 2C).
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } a^2 \sin 2B + b^2 \sin 2A &= 2a \sin B \cdot a \cos B + 2b \sin A \cdot b \cos A \\
 &= 2a \sin B (a \cos B + b \cos A) \\
 &= 2ac \sin B = 4\Delta.
 \end{aligned}$$

$$\text{Hence } \Sigma a^3 \cos (B-C) = 12R\Delta = 3abc.$$

22. (1) We have, from the given equation,

$$(b \sin \theta - c)^2 = a^2 \cos^2 \theta = a^2 (1 - \sin^2 \theta),$$

or

$$(a^2 + b^2) \sin^2 \theta - 2bc \sin \theta + (c^2 - a^2) = 0;$$

and this is the required equation since by hypothesis it is satisfied by $\sin \alpha$ and $\sin \beta$.

(2) The required equation is

$$x^2 - (\cos 2\alpha + \cos 2\beta)x + \cos 2\alpha \cos 2\beta = 0.$$

Now

$$\begin{aligned}
 \cos 2\alpha + \cos 2\beta &= 2 \cos (\alpha + \beta) \cos (\alpha - \beta) \\
 &= \frac{2(a^2 - b^2)}{a^2 + b^2} \cdot \frac{2c^2 - a^2 - b^2}{a^2 + b^2}.
 \end{aligned}$$

[See solutions to examples 4 and 6 above.]

Again,

$$\begin{aligned}
 \cos 2\alpha \cos 2\beta &= \cos^2 (\alpha - \beta) - \sin^2 (\alpha + \beta) \\
 &= \frac{(2c^2 - a^2 - b^2)^2 - 4a^2b^2}{(a^2 + b^2)^2}.
 \end{aligned}$$

Hence, by substitution the required equation is obtained.

Otherwise. Let $\cos \alpha = y$, then the given equation may be written

$$(ay - c)^2 = b^2(1 - y^2),$$

or

$$(a^2 + b^2)y^2 - b^2 + c^2 = 2acy \dots\dots\dots(1).$$

If

$$x = \cos 2\alpha = 2 \cos^2 \alpha - 1 = 2y^2 - 1,$$

we have

$$y^2 = \frac{x+1}{2}.$$

Substituting in (1) we obtain the equation

$$\left\{ (a^2 + b^2) \frac{x+1}{2} - b^2 + c^2 \right\}^2 = 4a^2c^2 \cdot \left(\frac{x+1}{2} \right),$$

which reduces to

$$\begin{aligned}
 (a^2 + b^2)^2 x^2 - 2(a^2 - b^2)(2c^2 - a^2 - b^2)x \\
 + (a^4 + b^4 + 4c^4 - 2a^2b^2 - 4a^2c^2 - 4b^2c^2) = 0.
 \end{aligned}$$

EXAMPLES. XXIV. b. PAGE 307.

$$\begin{aligned}
 1. \quad \Sigma \sin (\alpha - \theta) \sin (\beta - \gamma) &= \Sigma (\sin \alpha \cos \theta - \cos \alpha \sin \theta) \sin (\beta - \gamma) \\
 &= \cos \theta \Sigma \sin \alpha \sin (\beta - \gamma) - \sin \theta \Sigma \cos \alpha \sin (\beta - \gamma) \\
 &= 0.
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \Sigma (\cos \beta \cos \gamma - \sin \beta \sin \gamma) \sin (\beta - \gamma) &= \Sigma \cos (\beta + \gamma) \sin (\beta - \gamma) \\
 &= \frac{1}{2} \Sigma (\sin 2\beta - \sin 2\gamma) \\
 &= 0.
 \end{aligned}$$

$$\therefore \Sigma \cos \beta \cos \gamma \sin (\beta - \gamma) = \Sigma \sin \beta \sin \gamma \sin (\beta - \gamma).$$

$$\begin{aligned}
 3. \quad \Sigma \sin (\beta - \gamma) \cos (\beta + \gamma + \theta) \\
 &= \Sigma \sin (\beta - \gamma) \{ \cos \theta \cos (\beta + \gamma) - \sin \theta \sin (\beta + \gamma) \} \\
 &= \cos \theta \Sigma \sin (\beta - \gamma) \cos (\beta + \gamma) - \sin \theta \Sigma \sin (\beta - \gamma) \sin (\beta + \gamma) \\
 &= \frac{1}{2} \cos \theta \Sigma (\sin 2\beta - \sin 2\gamma) - \frac{1}{2} \sin \theta \Sigma (\cos 2\gamma - \cos 2\beta) \\
 &= 0.
 \end{aligned}$$

$$\begin{aligned}
 4. \quad \cos 2 (\beta - \gamma) + \cos 2 (\gamma - \alpha) + \cos 2 (\alpha - \beta) \\
 &= 2 \cos (\beta - \alpha) \cos (\alpha + \beta - 2\gamma) + 2 \cos^2 (\alpha - \beta) - 1 \\
 &= 2 \cos (\alpha - \beta) \{ \cos (\alpha + \beta - 2\gamma) + \cos (\alpha - \beta) \} - 1 \\
 &= 4 \cos (\alpha - \beta) \cos (\alpha - \gamma) \cos (\beta - \gamma) - 1.
 \end{aligned}$$

$$\begin{aligned}
 5. \quad \Sigma \sin \beta \sin \gamma \sin (\beta - \gamma) \\
 &= \frac{1}{2} \Sigma \{ \cos (\beta - \gamma) - \cos (\beta + \gamma) \} \sin (\beta - \gamma) \\
 &= \frac{1}{4} \Sigma \sin 2 (\beta - \gamma) - \frac{1}{4} \Sigma (\sin 2\beta - \sin 2\gamma) \\
 &= -\Pi \sin (\beta - \gamma).
 \end{aligned}$$

[Art. 306, Ex. 2.]

$$\begin{aligned}
 6. \quad \cot (\alpha - \beta) &= \cot \{ (\alpha - \gamma) - (\beta - \gamma) \} \\
 &= \frac{\cot (\alpha - \gamma) \cot (\beta - \gamma) + 1}{\cot (\beta - \gamma) - \cot (\alpha - \gamma)};
 \end{aligned}$$

by multiplying up and transposing we obtain the required result.

$$7. \quad 2\Sigma \sin 3a \sin (\beta - \gamma) = \cos (3a - \beta + \gamma) - \cos (3a + \beta - \gamma) + \cos (3\beta - \gamma + a) \\ - \cos (3\beta + \gamma - a) + \cos (3\gamma - a + \beta) - \cos (3\gamma + a - \beta).$$

Combining the first and fourth terms, the second and fifth terms, the third and sixth terms, and dividing by 2, we obtain

$$\Sigma \sin 3a \sin (\beta - \gamma) = \sin (a + \beta + \gamma) \{ \sin 2 (\beta - a) + \sin 2 (a - \gamma) + \sin 2 (\gamma - \beta) \} \\ = 4 \sin (a + \beta + \gamma) \Pi \sin (\beta - \gamma). \quad [\text{Art. 306, Ex. 2.}]$$

$$8. \quad 4\Sigma \cos^3 a \sin (\beta - \gamma) = \Sigma (\cos 3a + 3 \cos a) \sin (\beta - \gamma) \\ = \Sigma \cos 3a \sin (\beta - \gamma) \\ = 4 \cos (a + \beta + \gamma) \Pi \sin (\beta - \gamma). \quad [\text{Art. 306, Ex. 4.}]$$

$$9. \quad 4\Sigma \cos (\theta + a) \sin (\theta - a) \cos (\beta + \gamma) \sin (\beta - \gamma) \\ = \Sigma (\sin 2\theta - \sin 2a) (\sin 2\beta - \sin 2\gamma) \\ = \sin 2\theta \Sigma (\sin 2\beta - \sin 2\gamma) - \Sigma \sin 2a (\sin 2\beta - \sin 2\gamma) \\ = 0.$$

10. In the identity $\Sigma bc (b - c) = -\Pi (b - c)$, put

$$a = \sin^2 \alpha, \quad b = \sin^2 \beta, \quad c = \sin^2 \gamma;$$

then $b - c = \sin^2 \beta - \sin^2 \gamma = \sin (\beta + \gamma) \sin (\beta - \gamma);$

$$\therefore \Sigma \sin^2 \beta \sin^2 \gamma \sin (\beta + \gamma) \sin (\beta - \gamma) = -\Pi \sin (\beta + \gamma) \cdot \Pi \sin (\beta - \gamma).$$

11. In the identity $\Sigma bc (b - c) = -\Pi (b - c)$, put

$$a = \cos 2\alpha, \quad b = \cos 2\beta, \quad c = \cos 2\gamma;$$

then $b - c = \cos 2\beta - \cos 2\gamma = -2 \sin (\beta + \gamma) \sin (\beta - \gamma);$

$$\therefore -2\Sigma \cos 2\beta \cos 2\gamma \sin (\beta + \gamma) \sin (\beta - \gamma) = 8\Pi \sin (\beta + \gamma) \cdot \Pi \sin (\beta - \gamma).$$

12. In the identity $\Sigma a^2 (b - c) = -\Pi (b - c)$, put

$$a = \cos 2\alpha, \quad b = \cos 2\beta, \quad c = \cos 2\gamma;$$

then $b - c = \cos 2\beta - \cos 2\gamma = -2 \sin (\beta + \gamma) \sin (\beta - \gamma);$

$$\therefore \Sigma 2 \cos^2 2\alpha \sin (\beta + \gamma) \sin (\beta - \gamma) = -8\Pi \sin (\beta + \gamma) \cdot \Pi \sin (\beta - \gamma).$$

Also $\Sigma \sin (\beta + \gamma) \sin (\beta - \gamma) = 0;$

whence by subtraction, we have

$$\Sigma (2 \cos^2 2\alpha - 1) \sin (\beta + \gamma) \sin (\beta - \gamma) = -8\Pi \sin (\beta + \gamma) \cdot \Pi \sin (\beta - \gamma).$$

13. In the identity $\Sigma a^3(b-c) = -(a+b+c) \Pi(b-c)$, put

$$a = \sin \alpha, \quad b = \sin \beta, \quad c = \sin \gamma;$$

$$\therefore \Sigma \sin^3 \alpha (\sin \beta - \sin \gamma) = -(\sin \alpha + \sin \beta + \sin \gamma) \Pi (\sin \beta - \sin \gamma) \dots (1).$$

But $\Sigma \sin \alpha (\sin \beta - \sin \gamma) = 0 \dots \dots \dots (2);$

multiply (2) by 3, and (1) by 4; then by subtraction we obtain

$$\Sigma (3 \sin \alpha - 4 \sin^3 \alpha) (\sin \beta - \sin \gamma) = 4 (\sin \alpha + \sin \beta + \sin \gamma) \Pi (\sin \beta - \sin \gamma).$$

14. If $a+b+c=0$, then $a^3+b^3+c^3=3abc$.

The condition $a+b+c=0$ is satisfied, if $a = \sin(\beta+\gamma) \sin(\beta-\gamma)$, and b and c are equal to corresponding quantities.

15. The condition $a+b+c=0$ is satisfied, if $a = \cos(\beta+\gamma+\theta) \sin(\beta-\gamma)$, and b and c are equal to corresponding quantities.

16. We proceed exactly as in Art. 309, and shew that

$$a + \beta + \gamma = n\pi;$$

$$\therefore 3a + 3\beta + 3\gamma = 3n\pi.$$

From this relation it is easy to shew that

$$\Sigma \tan 3a = \Pi \tan 3a;$$

$$\therefore \Sigma \frac{3x - x^3}{1 - 3x^2} = \Pi \frac{3x - x^3}{1 - 3x^2}.$$

17. Put $x = \cot \alpha$, $y = \cot \beta$, $z = \cot \gamma$; then

$$\cot \beta \cot \gamma + \cot \gamma \cot \alpha + \cot \alpha \cot \beta = 1;$$

$$\therefore \cot \alpha = -\frac{\cot \beta \cot \gamma - 1}{\cot \gamma + \cot \beta} = -\cot(\beta + \gamma);$$

$$\therefore \alpha = n\pi - (\beta + \gamma), \text{ or } \alpha + \beta + \gamma = n\pi;$$

$$\therefore 2a + 2\beta + 2\gamma = 2n\pi.$$

From this relation it is easy to shew that

$$\cot 2\beta \cot 2\gamma + \cot 2\gamma \cot 2a + \cot 2a \cot 2\beta = 1;$$

$$\therefore \frac{(y^2 - 1)(z^2 - 1)}{4yz} + \frac{(z^2 - 1)(x^2 - 1)}{4zx} + \frac{(x^2 - 1)(y^2 - 1)}{4xy} = 1,$$

$$\therefore \Sigma x(1 - y^2)(1 - z^2) = 4xyz.$$

EXAMPLES. XXIV. c. PAGE 311.

1. If $A + B + C = 0,$

then $\cot C = -\cot(A + B) = -\frac{\cot A \cot B - 1}{\cot B + \cot A};$

that is, $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$

The given condition is satisfied if

$$A = 2\beta + \gamma - 3\alpha, \quad B = 2\gamma + \alpha - 3\beta, \quad C = 2\alpha + \beta - 3\gamma.$$

2. (1) In Example 4, Art. 133, we have proved that

$$4 \sin \alpha \sin \beta \sin \gamma = \Sigma \sin(\beta + \gamma - \alpha) - \sin(\alpha + \beta + \gamma).$$

In this identity first replace α, β, γ by $\beta + \gamma, \gamma + \alpha, \alpha + \beta$ respectively, and secondly replace α, β, γ by $2\alpha, 2\beta, 2\gamma$ respectively.

Thus $8\Pi \sin(\beta + \gamma) = 2 \sin 2\alpha - 2 \sin 2(\alpha + \beta + \gamma),$

and $4\Pi \sin 2\alpha = \Sigma \sin 2(\beta + \gamma - \alpha) - \sin 2(\alpha + \beta + \gamma).$

$$\therefore 8\Pi \sin(\beta + \gamma) + 4\Pi \sin 2\alpha$$

$$= 2\Sigma \sin 2\alpha + \Sigma \sin 2(\beta + \gamma - \alpha) - 3 \sin 2(\alpha + \beta + \gamma)$$

$$= 2\Sigma \sin 2\alpha + \Sigma \{\sin 2(\beta + \gamma - \alpha) - \sin 2(\alpha + \beta + \gamma)\}$$

$$= 2\Sigma \sin 2\alpha - 2\Sigma \cos 2(\beta + \gamma) \sin 2\alpha$$

$$= 2\Sigma \sin 2\alpha \{1 - \cos 2(\beta + \gamma)\}$$

$$= 4\Sigma \sin 2\alpha \sin^2(\beta + \gamma).$$

(2) In the first part of this Example, we have seen that

$$8\Pi \sin \alpha = 2\Sigma \sin(\beta + \gamma - \alpha) - 2 \sin(\alpha + \beta + \gamma).$$

By replacing α, β, γ by $\beta + \gamma - \alpha, \gamma + \alpha - \beta, \alpha + \beta - \gamma$ respectively, we have

$$4\Pi \sin(\beta + \gamma - \alpha) = \Sigma \sin(3\alpha - \beta - \gamma) - \sin(\alpha + \beta + \gamma).$$

$$\therefore 4\Pi \sin(\beta + \gamma - \alpha) + 8\Pi \sin \alpha$$

$$= 2\Sigma \sin(\beta + \gamma - \alpha) + \Sigma \sin(3\alpha - \beta - \gamma) - 3 \sin(\alpha + \beta + \gamma)$$

$$= 2\Sigma \sin(\beta + \gamma - \alpha) + \Sigma \{\sin(3\alpha - \beta - \gamma) - \sin(\alpha + \beta + \gamma)\}$$

$$= 2\Sigma \sin(\beta + \gamma - \alpha) - 2\Sigma \cos 2\alpha \sin(\beta + \gamma - \alpha)$$

$$= 2\Sigma \sin(\beta + \gamma - \alpha) \{1 - \cos 2\alpha\}$$

$$= 4\Sigma \sin^2 \alpha \sin(\beta + \gamma - \alpha).$$

3. (1) This is equivalent to proving that in the pedal triangle

$$a'^2 - b'^2 = 2R'c' \sin (A' - B').$$

Now in *any* triangle

$$\begin{aligned} a^2 - b^2 &= 4R^2 (\sin^2 A - \sin^2 B) \\ &= 4R^2 \sin (A + B) \sin (A - B) \\ &= 2R \cdot 2R \sin C \cdot \sin (A - B) \\ &= 2Rc \sin (A - B). \end{aligned}$$

(2) This is equivalent to proving that in the ex-central triangle

$$a_1^2 - b_1^2 = 2R_1c_1 \sin (A_1 - B_1).$$

This identity has been proved in (1).

(3) This is equivalent to proving that in the pedal triangle

$$\Sigma (b' + c') \tan \frac{A'}{2} = 4R' \Sigma \cos A'.$$

Now in *any* triangle

$$\begin{aligned} \Sigma (b + c) \tan \frac{A}{2} &= 2R \Sigma (\sin B + \sin C) \tan \frac{A}{2} \\ &= 4R \Sigma \sin \frac{B+C}{2} \cos \frac{B-C}{2} \sin \frac{A}{2} \div \cos \frac{A}{2} \\ &= 4R \Sigma \cos \frac{B-C}{2} \cos \frac{B+C}{2} \\ &= 2R \Sigma (\cos B + \cos C) \\ &= 4R (\cos A + \cos B + \cos C). \end{aligned}$$

4. We have $\sin^2 2\theta = 4 \sin^2 \alpha \sin^2 \gamma = (1 - \cos 2\alpha) (1 - \cos 2\gamma)$;

$$\therefore 1 - \cos^2 2\theta = \left(1 - \frac{\cos 2\theta}{\cos 2\beta}\right) \left(1 - \frac{\cos 2\theta}{\cos 2\delta}\right);$$

$$\therefore \cos 2\theta \left(\frac{1}{\cos 2\beta} + \frac{1}{\cos 2\delta}\right) = \cos^2 2\theta \left(1 + \frac{1}{\cos 2\beta \cos 2\delta}\right);$$

$$\therefore \cos 2\theta = \frac{\cos 2\beta + \cos 2\delta}{1 + \cos 2\beta \cos 2\delta};$$

$$\therefore \frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \frac{(1 - \cos 2\beta)(1 - \cos 2\delta)}{(1 + \cos 2\beta)(1 + \cos 2\delta)};$$

$$\therefore \tan^2 \theta = \tan^2 \beta \tan^2 \delta.$$

5. We have $\tan^2 \frac{\gamma}{2} = \tan^2 \frac{\theta}{2} \tan^2 \frac{\phi}{2}$;

$$\begin{aligned}\therefore \frac{1 - \cos \gamma}{1 + \cos \gamma} &= \frac{1 - \cos \theta}{1 + \cos \theta} \cdot \frac{1 - \cos \phi}{1 + \cos \phi} \\ &= \frac{1 - \cos \alpha \cos \gamma}{1 + \cos \alpha \cos \gamma} \cdot \frac{1 - \cos \beta \cos \gamma}{1 + \cos \beta \cos \gamma}.\end{aligned}$$

Componendo and Dividendo,

$$\begin{aligned}\frac{1}{\cos \gamma} &= \frac{1 + \cos \alpha \cos \beta \cos^2 \gamma}{(\cos \alpha + \cos \beta) \cos \gamma}; \\ \therefore \cos^2 \gamma &= \frac{\cos \alpha + \cos \beta - 1}{\cos \alpha \cos \beta}; \\ \therefore \sin^2 \gamma &= \frac{(1 - \cos \alpha)(1 - \cos \beta)}{\cos \alpha \cos \beta} = (\sec \alpha - 1)(\sec \beta - 1).\end{aligned}$$

6. By solving for $\cos \theta$, we have

$$\cos \theta = \frac{\sin^2 \beta \cos^2 \alpha - \sin^2 \alpha \cos^2 \beta}{\sin^2 \beta \cos \alpha - \sin^2 \alpha \cos \beta}.$$

By substituting $1 - \cos^2 \beta$ for $\sin^2 \beta$, and $1 - \cos^2 \alpha$ for $\sin^2 \alpha$, we have

$$\begin{aligned}\cos \theta &= \frac{\cos^2 \alpha - \cos^2 \beta}{(1 - \cos^2 \beta) \cos \alpha - (1 - \cos^2 \alpha) \cos \beta} \\ &= \frac{\cos \alpha + \cos \beta}{1 + \cos \alpha \cos \beta}. \\ \therefore \frac{1 - \cos \theta}{1 + \cos \theta} &= \frac{(1 - \cos \alpha)(1 - \cos \beta)}{(1 + \cos \alpha)(1 + \cos \beta)}; \\ \therefore \tan^2 \frac{\theta}{2} &= \tan^2 \frac{\alpha}{2} \tan^2 \frac{\beta}{2}.\end{aligned}$$

7. From the two given equations, we have

$$\begin{aligned}\sec \alpha &= \frac{\tan \beta}{\tan \theta}, \text{ and } \tan \alpha = \frac{\tan \gamma}{\sin \theta}; \\ \therefore \frac{\tan^2 \beta}{\tan^2 \theta} - \frac{\tan^2 \gamma}{\sin^2 \theta} &= 1; \\ \therefore \frac{\tan^2 \beta \cos^2 \theta}{1 - \cos^2 \theta} - \frac{\tan^2 \gamma}{1 - \cos^2 \theta} &= 1; \\ \therefore \tan^2 \beta \cos^2 \theta - \tan^2 \gamma &= 1 - \cos^2 \theta; \\ \therefore \cos^2 \theta &= \frac{1 + \tan^2 \gamma}{1 + \tan^2 \beta} = \frac{\sec^2 \gamma}{\sec^2 \beta}.\end{aligned}$$

8. We have $bc \cos \alpha \cos \phi + ac \sin \alpha \sin \phi - ab = 0$,
and $bc \cos \beta \cos \phi + ac \sin \beta \sin \phi - ab = 0$;

whence by cross multiplication

$$\frac{\cos \phi}{a^2 bc (\sin \beta - \sin \alpha)} = \frac{\sin \phi}{ab^2 c (\cos \alpha - \cos \beta)} = \frac{1}{abc^2 \sin (\beta - \alpha)};$$

$$\therefore \frac{\cos \phi}{a \cos \frac{\alpha + \beta}{2}} = \frac{\sin \phi}{b \sin \frac{\alpha + \beta}{2}} = \frac{1}{c \cos \frac{\alpha - \beta}{2}};$$

$$\therefore a^2 \cos^2 \frac{\alpha + \beta}{2} + b^2 \sin^2 \frac{\alpha + \beta}{2} = c^2 \cos^2 \frac{\alpha - \beta}{2};$$

$$\therefore a^2 \{1 + \cos (\alpha + \beta)\} + b^2 \{1 - \cos (\alpha + \beta)\} = c^2 \{1 + \cos (\alpha - \beta)\};$$

$$\therefore (b^2 + c^2 - a^2) \cos \alpha \cos \beta + (c^2 + a^2 - b^2) \sin \alpha \sin \beta = a^2 + b^2 - c^2.$$

9. From the given equation, we have

$$\sin^2 \alpha (\cos \theta - \cos \alpha)^2 = \cos^2 \alpha \sin^2 \theta = \cos^2 \alpha (1 - \cos^2 \theta);$$

$$\therefore \cos^2 \theta - 2 \cos \alpha \sin^2 \alpha \cos \theta - \cos^4 \alpha = 0;$$

which is a quadratic in $\cos \theta$ with roots $\cos \beta$ and $\cos \gamma$.

$$\therefore \cos \beta \cos \gamma = \cos^4 \alpha.$$

Similarly, from the equation

$$\cos^2 \alpha (\sin \theta - \sin \alpha)^2 = \sin^2 \alpha \cos^2 \theta = \sin^2 \alpha (1 - \sin^2 \theta),$$

we may shew that $\sin \beta \sin \gamma = \sin^4 \alpha$.

$$\therefore \frac{\cos \beta \cos \gamma}{\cos^2 \alpha} + \frac{\sin \beta \sin \gamma}{\sin^2 \alpha} = \cos^2 \alpha + \sin^2 \alpha = 1.$$

10. We have $k^2 \cos \beta \cos \alpha + k \sin \alpha + (k \sin \beta + 1) = 0$ (1),

and $k^2 \cos \gamma \cos \alpha + k \sin \alpha + (k \sin \gamma + 1) = 0$ (2);

whence by cross multiplication,

$$\frac{\cos \alpha}{k^2 (\sin \beta - \sin \gamma)} = \frac{\sin \alpha}{k^3 \sin (\beta - \gamma) + k^2 (\cos \gamma - \cos \beta)} = \frac{1}{k^3 (\cos \beta - \cos \gamma)};$$

$$\therefore \frac{\cos \alpha}{\cos \frac{\beta + \gamma}{2}} = \frac{\sin \alpha}{k \cos \frac{\beta - \gamma}{2} + \sin \frac{\beta + \gamma}{2}} = -\frac{1}{k \sin \frac{\beta + \gamma}{2}};$$

$$\therefore \cos^2 \frac{\beta + \gamma}{2} + \left\{ k \cos \frac{\beta - \gamma}{2} + \sin \frac{\beta + \gamma}{2} \right\}^2 = k^2 \sin^2 \frac{\beta + \gamma}{2};$$

$$\therefore k^2 \left(\cos^2 \frac{\beta - \gamma}{2} - \sin^2 \frac{\beta + \gamma}{2} \right) + k (\sin \beta + \sin \gamma) + 1 = 0;$$

$$\therefore k^2 \cos \beta \cos \gamma + k (\sin \beta + \sin \gamma) + 1 = 0.$$

Otherwise. Form a quadratic equation in $\sin \theta$; thus

$$k^4 \cos^2 \alpha (1 - \sin^2 \theta) = (1 + k \sin \alpha + k \sin \theta)^2;$$

$\sin \beta + \sin \gamma =$ sum of roots of this equation

$$= - \frac{\text{coefficient of } \sin \theta}{\text{coefficient of } \sin^2 \theta}$$

$$= - \frac{2(1 + k \sin \alpha)k}{k^2(1 + k^2 \cos^2 \alpha)};$$

$$\therefore k(\sin \beta + \sin \gamma) = - \frac{2(1 + k \sin \alpha)}{1 + k^2 \cos^2 \alpha} \dots \dots \dots (1).$$

Again, form a quadratic equation in $\cos \theta$; thus

$$k^2(1 - \cos^2 \theta) = (1 + k \sin \alpha + k^2 \cos \alpha \cos \theta)^2;$$

$\cos \beta \cos \gamma =$ product of roots of this equation

$$= \frac{k^2 - (1 + k \sin \alpha)^2}{-k^2 - k^4 \cos^2 \alpha};$$

$$\therefore k^2 \cos \beta \cos \gamma = \frac{1 + 2k \sin \alpha - k^2 \cos^2 \alpha}{1 + k^2 \cos^2 \alpha} \dots \dots \dots (2).$$

By adding (1) and (2) we obtain the required result.

11. We have

$$\sin^2 \alpha \cos \beta \cos \phi + \cos^2 \alpha \sin \beta \sin \phi + \cos^2 \alpha \sin^2 \alpha = 0,$$

$$\text{and} \quad \sin^2 \alpha \cos \gamma \cos \phi + \cos^2 \alpha \sin \gamma \sin \phi + \cos^2 \alpha \sin^2 \alpha = 0;$$

whence by cross multiplication

$$\frac{\cos \phi}{\cos^4 \alpha \sin^2 \alpha (\sin \beta - \sin \gamma)} = \frac{\sin \phi}{\cos^2 \alpha \sin^4 \alpha (\cos \gamma - \cos \beta)}$$

$$= \frac{1}{\sin^2 \alpha \cos^2 \alpha \sin(\gamma - \beta)};$$

$$\therefore \frac{\cos \phi}{\cos^2 \alpha \cos \frac{\beta + \gamma}{2}} = \frac{\sin \phi}{\sin^2 \alpha \sin \frac{\beta + \gamma}{2}} = - \frac{1}{\cos \frac{\beta - \gamma}{2}};$$

$$\therefore \cos^4 \alpha \cos^2 \frac{\beta + \gamma}{2} + \sin^4 \alpha \sin^2 \frac{\beta + \gamma}{2} = \cos^2 \frac{\beta - \gamma}{2};$$

$$\therefore \cos^4 \alpha \{1 + \cos(\beta + \gamma)\} + \sin^4 \alpha \{1 - \cos(\beta + \gamma)\} = 1 + \cos(\beta - \gamma);$$

$$\therefore \cos \beta \cos \gamma (1 - \cos^4 \alpha + \sin^4 \alpha) + \sin \beta \sin \gamma (1 + \cos^4 \alpha - \sin^4 \alpha)$$

$$= \cos^4 \alpha + \sin^4 \alpha - 1.$$

Writing $\cos^4 \alpha + \sin^4 \alpha + 2 \sin^2 \alpha \cos^2 \alpha$ instead of unity we have

$$2 \cos \beta \cos \gamma \sin^2 \alpha (\cos^2 \alpha + \sin^2 \alpha) + 2 \sin \beta \sin \gamma \cos^2 \alpha (\cos^2 \alpha + \sin^2 \alpha)$$

$$= -2 \sin^2 \alpha \cos^2 \alpha;$$

$$\text{that is,} \quad \sin^2 \alpha \cos \beta \cos \gamma + \cos^2 \alpha \sin \beta \sin \gamma + \sin^2 \alpha \cos^2 \alpha = 0.$$

EXAMPLES. XXV. a. PAGE 318.

$$1. \quad p \cot \theta + q \tan \theta = (\sqrt{p \cot \theta} - \sqrt{q \tan \theta})^2 + 2 \sqrt{pq}.$$

$$2. \quad 4 \sin^2 \theta + \operatorname{cosec}^2 \theta = (2 \sin \theta - \operatorname{cosec} \theta)^2 + 4.$$

$$3. \quad 8 \sec^2 \theta + 18 \cos^2 \theta = 2 \{ (2 \sec \theta - 3 \cos \theta)^2 + 12 \}.$$

$$4. \quad 3 - 2 \cos \theta + \cos^2 \theta = 2 + (1 - \cos \theta)^2.$$

5. We have $\tan^2 \beta + \tan^2 \gamma > 2 \tan \beta \tan \gamma$, and two corresponding inequalities. [See Art. 316.]

$$6. \quad \text{Since } (1 - \sin \alpha)^2 \text{ is positive, } 1 + \sin^2 \alpha > 2 \sin \alpha.$$

$$\text{Similarly, } 1 + \sin^2 \beta > 2 \sin \beta.$$

$$\therefore 2 + \sin^2 \alpha + \sin^2 \beta > 2 \sin \alpha + 2 \sin \beta.$$

$$7. \quad \sin \theta + \cos \theta = \sqrt{2} \sin \left(\theta + \frac{\pi}{4} \right).$$

$$8. \quad \cos \theta + \sqrt{3} \sin \theta = 2 \sin \left(\theta + \frac{\pi}{6} \right).$$

$$9. \quad a \cos (\alpha + \theta) + b \sin \theta = a \cos \alpha \cos \theta + (b - a \sin \alpha) \sin \theta.$$

$$\therefore \text{maximum value} = \sqrt{a^2 \cos^2 \alpha + (b - a \sin \alpha)^2}. \quad [\text{Art. 317.}]$$

$$10. \quad p \cos \theta + q \sin (\alpha + \theta) = (p + q \sin \alpha) \cos \theta + q \cos \alpha \sin \theta.$$

$$\therefore \text{maximum value} = \sqrt{(p + q \sin \alpha)^2 + q^2 \cos^2 \alpha}.$$

$$11. \quad \sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} = 2 \sin \frac{\sigma}{2} \cos \frac{\alpha - \beta}{2}.$$

$$\therefore \text{maximum value} = 2 \sin \frac{\sigma}{2}.$$

$$12. \quad \sin \alpha \sin \beta = \frac{1}{2} \{ \cos (\alpha - \beta) - \cos (\alpha + \beta) \} = \frac{1}{2} \{ \cos (\alpha - \beta) - \cos \sigma \}.$$

$$\therefore \text{maximum value} = \frac{1}{2} (1 - \cos \sigma) = \sin^2 \frac{\sigma}{2}.$$

$$13. \quad \tan \alpha + \tan \beta = \frac{\sin (\alpha + \beta)}{\cos \alpha \cos \beta} = \frac{\sin \sigma}{\cos \alpha \cos \beta}.$$

By Art. 319, the denominator is a maximum when $\alpha = \beta$, and in this case $\tan \alpha + \tan \beta$ is a minimum, its value being $2 \tan \frac{\sigma}{2}$.

$$\begin{aligned}
 14. \quad \operatorname{cosec} a + \operatorname{cosec} \beta &= \frac{\sin a + \sin \beta}{\sin a \sin \beta} \\
 &= \frac{4 \sin \frac{a+\beta}{2} \cos \frac{a-\beta}{2}}{\cos(a-\beta) - \cos(a+\beta)} = \frac{2 \sin \frac{a+\beta}{2} \cos \frac{a-\beta}{2}}{\cos^2 \frac{a-\beta}{2} - \cos^2 \frac{a+\beta}{2}} \\
 &= \sin \frac{a+\beta}{2} \left(\frac{1}{\cos \frac{a-\beta}{2} + \cos \frac{a+\beta}{2}} + \frac{1}{\cos \frac{a-\beta}{2} - \cos \frac{a+\beta}{2}} \right).
 \end{aligned}$$

Since $a + \beta$ is constant, this expression is least when the denominators are greatest, that is, when $a = \beta = \frac{\sigma}{2}$.

Thus the minimum value is $2 \operatorname{cosec} \frac{\sigma}{2}$.

$$\begin{aligned}
 15. \quad \cos A \cos B \cos C &= \frac{1}{2} \cos C \{ \cos(A-B) + \cos(A+B) \} \\
 &= \frac{1}{2} \cos C \{ \cos(A-B) - \cos C \}.
 \end{aligned}$$

Supposing C constant, this expression is not a maximum unless $A = B$. Similarly, we may shew that the given expression is not a maximum unless $A = B = C = 60^\circ$. In this case its value is $\cos^3 60^\circ$ or $\frac{1}{8}$.

$$16. \quad \cot A + \cot B + \cot C = \frac{\sin(A+B)}{\sin A \sin B} + \cot C = \frac{\sin C}{\sin A \sin B} + \cot C.$$

Supposing C constant, $\sin A \sin B$ is a maximum when $A = B$, and in this case the given expression is a minimum. Thus the given expression is a minimum when $A = B = C$, and its value is $3 \cot 60^\circ$ or $\sqrt{3}$.

$$\begin{aligned}
 17. \quad \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} &= \frac{1}{2} (1 - \cos A) + \dots + \dots \\
 &= \frac{3}{2} - \frac{1}{2} (\cos A + \cos B + \cos C).
 \end{aligned}$$

As in Example 1, p. 315, it is easy to shew that $\cos A + \cos B + \cos C$ is a maximum when $A = B = C$, and in this case the given expression is a minimum, its value being $3 \sin^2 \frac{60^\circ}{2}$ or $\frac{3}{4}$.

18. If C is constant, $A + B$ is constant, and therefore $\sec A + \sec B$ is a minimum when $A = B$. [See Ex. 2, p. 316.]

Thus the given expression is a minimum when $A = B = C$, its value being $3 \sec 60^\circ$ or 6.

$$\begin{aligned}
 19. \quad \tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \\
 &= \left(\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right)^2 - 2 \sum \tan \frac{B}{2} \tan \frac{C}{2} \\
 &= \left(\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right)^2 - 2.
 \end{aligned}$$

As in Example 16, it is easy to shew that $\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2}$ is a minimum when $A=B=C$, and in this case the given expression is a minimum, its value being $3 \tan^2 \frac{60^\circ}{2}$ or 1.

$$\begin{aligned}
 20. \quad \cot^2 A + \cot^2 B + \cot^2 C &= (\cot A + \cot B + \cot C)^2 - 2 \sum \cot B \cot C \\
 &= (\cot A + \cot B + \cot C)^2 - 2.
 \end{aligned}$$

But it has been shewn in Example 16 that the right side is a minimum when $A=B=C$, and in this case the given expression is a minimum, its value being $3 \cot^2 60^\circ$ or 1.

$$\begin{aligned}
 21. \quad 2(a \sin^2 \theta + b \sin \theta \cos \theta + c \cos^2 \theta) \\
 &= a(1 - \cos 2\theta) + b \sin 2\theta + c(1 + \cos 2\theta) \\
 &= a + c + b \sin 2\theta - (a - c) \cos 2\theta \dots\dots\dots(1).
 \end{aligned}$$

The greatest value of $b \sin 2\theta - (a - c) \cos 2\theta$ is $\sqrt{b^2 + (a - c)^2}$, which is less than $\sqrt{4ac + (a - c)^2}$ or $a + c$, since $b^2 < 4ac$.

Hence as in Art. 317, the maximum and minimum values of (1) are

$$a + c \pm \sqrt{b^2 + (a - c)^2}.$$

$$\begin{aligned}
 22. \quad \sin \alpha + \sin \beta + \sin \gamma - \sin(\alpha + \beta + \gamma) \\
 &= 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} - 2 \cos \frac{\alpha + \beta + 2\gamma}{2} \sin \frac{\alpha + \beta}{2} \\
 &= 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\gamma + \alpha}{2} \sin \frac{\beta + \gamma}{2}.
 \end{aligned}$$

The expression on the right is positive, since each of its component factors is positive.

$$\therefore \sin \alpha + \sin \beta + \sin \gamma > \sin(\alpha + \beta + \gamma).$$

23. Let $x = a \operatorname{cosec} \theta - b \cot \theta$, and put $\cot \theta = t$;

then

$$x = a \sqrt{1 + t^2} - bt.$$

As in Ex. 2, page 317, we may shew that $x > \sqrt{a^2 - b^2}$;

$$\therefore a \operatorname{cosec} \theta - b \cot \theta > \sqrt{a^2 - b^2}.$$

$$24. \text{ Let } x = \frac{\sec^2 \theta - \tan \theta}{\sec^2 \theta + \tan \theta} = \frac{1 - \tan \theta + \tan^2 \theta}{1 + \tan \theta + \tan^2 \theta};$$

$$\therefore \tan^2 \theta (x-1) + \tan \theta (x+1) + x-1 = 0.$$

In order that the values of $\tan \theta$ found from this quadratic may be real, we must have

$$(x+1)^2 > 4(x-1)^2;$$

that is, $-3x^2 + 10x - 3$ must be positive;

that is, $(3x-1)(x-3)$ must be negative.

Hence x must lie between 3 and $\frac{1}{3}$.

25. Denote the expression by x ; then

$$x = \frac{\tan^4 \theta + \tan^2 \theta - 1}{\tan^4 \theta - \tan^2 \theta + 1};$$

$$\therefore \tan^4 \theta (x-1) - \tan^2 \theta (x+1) + x+1 = 0.$$

In order that the values of $\tan^2 \theta$ found from this equation may be real, we must have

$$(x+1)^2 > 4(x+1)(x-1);$$

$$\therefore (x+1)(5-3x) > 0.$$

Thus the greatest value of x is $\frac{5}{3}$.

26. We have

$$\begin{aligned} & (a^2 + b^2 + c^2)(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) - (a \cos \alpha + b \cos \beta + c \cos \gamma)^2 \\ &= (b \cos \gamma - c \cos \beta)^2 + (c \cos \alpha - a \cos \gamma)^2 + (a \cos \beta - b \cos \alpha)^2, \end{aligned}$$

the minimum value of which is zero.

$$\therefore \text{minimum value of } (a^2 + b^2 + c^2)(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) - k^2 = 0.$$

$$\begin{aligned} \text{Again, } & (a+b+c)(a \cos^2 \alpha + b \cos^2 \beta + c \cos^2 \gamma) - (a \cos \alpha + b \cos \beta + c \cos \gamma)^2 \\ &= bc(\cos \beta - \cos \gamma)^2 + ca(\cos \gamma - \cos \alpha)^2 + ab(\cos \alpha - \cos \beta)^2, \end{aligned}$$

the minimum value of which is zero.

$$\therefore \text{minimum value of } (a+b+c)(a \cos^2 \alpha + b \cos^2 \beta + c \cos^2 \gamma) - k^2 = 0.$$

EXAMPLES. XXV. b. PAGE 324.

1. By squaring each of the given equations and adding, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2.$$

2. By transposition, we have

$$a \sec \theta = x \tan \theta + y, \quad b \sec \theta = x - y \tan \theta;$$

whence by squaring and adding,

$$(a^2 + b^2) \sec^2 \theta = (x^2 + y^2) (1 + \tan^2 \theta),$$

or

$$x^2 + y^2 = a^2 + b^2.$$

3. We have $\cos \theta + \sin \theta = a$, and $\cos^2 \theta - \sin^2 \theta = b$;

$$\therefore \cos \theta - \sin \theta = \frac{b}{a};$$

$$\therefore a^2 + \frac{b^2}{a^2} = 2.$$

4. We have $x^2 = (\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$,

and

$$y = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{2}{\sin 2\theta};$$

$$\therefore y(x^2 - 1) = 2.$$

5. By addition and subtraction, we have

$$\cot \theta = \frac{a+b}{2}, \text{ and } \cos \theta = \frac{a-b}{2};$$

whence by division,

$$\sin \theta = \frac{a-b}{a+b};$$

$$\therefore \left(\frac{a-b}{2}\right)^2 + \left(\frac{a-b}{a+b}\right)^2 = 1, \text{ or } \left(\frac{a-b}{2}\right)^2 = \frac{4ab}{(a+b)^2},$$

that is,

$$(a^2 - b^2)^2 = 16ab.$$

6. Here $x = \cot \theta + \frac{1}{\cot \theta} = \frac{\cot^2 \theta + 1}{\cot \theta} = \frac{\operatorname{cosec}^2 \theta}{\cot \theta},$

and

$$y = \operatorname{cosec} \theta - \frac{1}{\operatorname{cosec} \theta} = \frac{\operatorname{cosec}^2 \theta - 1}{\operatorname{cosec} \theta} = \frac{\cot^2 \theta}{\operatorname{cosec} \theta};$$

$$\therefore x^2 y = \operatorname{cosec}^3 \theta, \text{ and } xy^2 = \cot^3 \theta.$$

But

$$\operatorname{cosec}^2 \theta - \cot^2 \theta = 1;$$

$$\therefore x^{\frac{4}{3}} y^{\frac{2}{3}} - x^{\frac{2}{3}} y^{\frac{4}{3}} = 1.$$

7. Here

$$a^3 = \frac{1}{\sin \theta} - \sin \theta = \frac{\cos^2 \theta}{\sin \theta},$$

and

$$b^3 = \frac{1}{\cos \theta} - \cos \theta = \frac{\sin^2 \theta}{\cos \theta};$$

$$\therefore a^6 b^3 = \cos^3 \theta, \text{ or } a^2 b = \cos \theta;$$

and

$$a^3 b^6 = \sin^3 \theta, \text{ or } ab^2 = \sin \theta;$$

$$\therefore a^4 b^2 + a^2 b^4 = 1.$$

8. By substituting for $\cos 3\theta$ and $\sin 3\theta$, we have

$$4x = 4a \cos^3 \theta, \text{ or } x^{\frac{1}{3}} = a^{\frac{1}{3}} \cos \theta,$$

and

$$4y = 4a \sin^3 \theta, \text{ or } y^{\frac{1}{3}} = a^{\frac{1}{3}} \sin \theta;$$

$$\therefore x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

9. By transposition, we have

$$x(1 + \tan^2 \theta) = a \tan^3 \theta, \text{ or } x = \frac{a \tan^3 \theta}{\sec^2 \theta},$$

and

$$y(\sec^2 \theta - 1) = a \sec^3 \theta, \text{ or } y = \frac{a \sec^3 \theta}{\tan^2 \theta};$$

$$\therefore x^3 y^2 = a^5 \tan^5 \theta, \text{ and } x^2 y^3 = a^5 \sec^5 \theta.$$

$$\therefore (x^2 y^3)^{\frac{2}{5}} - (x^3 y^2)^{\frac{2}{5}} = a^2 (\sec^2 \theta - \tan^2 \theta) = a^2.$$

10. Here $x = a \cos \theta (2 \cos 2\theta - 1) = a \cos \theta (4 \cos^2 \theta - 3)$

$$= a \cos 3\theta.$$

Similarly

$$y = b \sin 3\theta.$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

11. Here

$$\sin \theta \sin a + \cos \theta \cos a = a,$$

and

$$\sin \theta \cos \beta - \cos \theta \sin \beta = b;$$

$$\therefore \sin \theta \cos (a - \beta) = a \sin \beta + b \cos a,$$

and

$$\cos \theta \cos (a - \beta) = a \cos \beta - b \sin a;$$

$$\therefore \cos^2 (a - \beta) = a^2 + b^2 - 2ab \sin (a - \beta).$$

12. Here

$$x + y = 3 - (1 - 2 \sin^2 2\theta) = 2 + 2 \sin^2 2\theta,$$

and

$$x - y = 4 \sin 2\theta;$$

$$\therefore x = 1 + 2 \sin 2\theta + \sin^2 2\theta = (1 + \sin 2\theta)^2,$$

or

$$x^{\frac{1}{2}} = 1 + \sin 2\theta.$$

Similarly,

$$y^{\frac{1}{2}} = 1 - \sin 2\theta.$$

$$\therefore x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2.$$

13. By addition and subtraction, we have

$$\begin{aligned}x + y &= (\sin \theta + \cos \theta) (1 + \sin 2\theta) \\&= (\sin \theta + \cos \theta)^3,\end{aligned}$$

and

$$\begin{aligned}x - y &= (\sin \theta - \cos \theta) (1 - \sin 2\theta) \\&= (\sin \theta - \cos \theta)^3;\end{aligned}$$

$$\begin{aligned}\therefore (x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} &= (\sin \theta + \cos \theta)^2 + (\sin \theta - \cos \theta)^2 \\&= 2.\end{aligned}$$

14. Here

$$a^2 = 1 + \sin 2\theta,$$

$$\therefore a^2 - 1 + \cos 2\theta = b.$$

$$\therefore \sin 2\theta = a^2 - 1, \text{ and } \cos 2\theta = -(a^2 - b - 1);$$

$$\therefore (a^2 - 1)^2 + (a^2 - b - 1)^2 = 1.$$

15. Here

$$\begin{aligned}a &= (4 \cos^3 \theta - 3 \cos \theta) + (3 \sin \theta - 4 \sin^3 \theta) \\&= (\cos \theta - \sin \theta) \{4 (\cos^2 \theta + \cos \theta \sin \theta + \sin^2 \theta) - 3\} \\&= b (1 + 2 \sin 2\theta).\end{aligned}$$

But

$$1 - \sin 2\theta = b^2;$$

$$\therefore a = b (1 + 2 - 2b^2) = 3b - 2b^3.$$

16. By squaring the first equation, and multiplying by 2, we have

$$a^2 (1 + \cos 2\theta) + b^2 (1 - \cos 2\theta) - 2ab \sin 2\theta = 2c^2;$$

$$\therefore (a^2 - b^2) \cos 2\theta - 2ab \sin 2\theta = 2c^2 - a^2 - b^2.$$

From the second equation,

$$(a^2 - b^2) \sin 2\theta + 2ab \cos 2\theta = 2c^2.$$

By squaring and adding, we obtain

$$(a^2 - b^2)^2 + 4a^2b^2 = 4c^4 - 4c^2 (a^2 + b^2) + (a^2 + b^2)^2 + 4c^4;$$

$$\therefore 0 = 8c^4 - 4c^2 (a^2 + b^2);$$

$$\therefore a^2 + b^2 = 2c^2.$$

17. By squaring and adding, we have

$$\begin{aligned}x^2 + y^2 &= a^2 + b^2 + 2ab (\cos \theta \cos 2\theta + \sin \theta \sin 2\theta) \\&= a^2 + b^2 + 2ab \cos \theta.\end{aligned}$$

Again,

$$x + b = a \cos \theta + 2b \cos^2 \theta = \cos \theta (a + 2b \cos \theta),$$

and

$$y = \sin \theta (a + 2b \cos \theta);$$

$$\therefore (x + b)^2 + y^2 = (a + 2b \cos \theta)^2$$

$$= \frac{1}{a^2} (x^2 + y^2 - b^2)^2.$$

18. Componendo and dividendo, we have

$$\frac{a}{b} = \frac{\tan(\theta + \alpha) + \tan(\theta - \alpha)}{\tan(\theta + \alpha) - \tan(\theta - \alpha)} = \frac{\sin 2\theta}{\sin 2\alpha};$$

$$\therefore b \sin 2\theta = a \sin 2\alpha.$$

Also

$$b \cos 2\theta = c - a \cos 2\alpha;$$

$$\therefore b^2 = c^2 - 2ac \cos 2\alpha + a^2.$$

19. By squaring and adding, we have

$$\begin{aligned} x^2 + y^2 &= a^2 \{2 - 2(\cos 3\theta \cos \theta + \sin 3\theta \sin \theta)\} \\ &= a^2 (2 - 2 \cos 2\theta) \\ &= 4a^2 \sin^2 \theta. \end{aligned}$$

And

$$2a^2 - x^2 - y^2 = 2a^2 \cos 2\theta.$$

But

$$x = 2a \cos 2\theta \sin \theta;$$

$$\begin{aligned} \therefore 4a^4 x^2 &= (2a^2 \cos 2\theta)^2 (4a^2 \sin^2 \theta) \\ &= (2a^2 - x^2 - y^2)^2 (x^2 + y^2). \end{aligned}$$

20. Solving the given equations for x and y , we have

$$\begin{aligned} x &= a (\cos \theta \cos 2\theta + 2 \sin \theta \sin 2\theta); \\ \therefore 2x &= a \{(\cos 3\theta + \cos \theta) + 2(\cos \theta - \cos 3\theta)\} \\ &= a (3 \cos \theta - \cos 3\theta) = a (6 \cos \theta - 4 \cos^3 \theta). \end{aligned}$$

And

$$\begin{aligned} y &= a (2 \sin 2\theta \cos \theta - \cos 2\theta \sin \theta); \\ \therefore 2y &= a \{2(\sin 3\theta + \sin \theta) - (\sin 3\theta - \sin \theta)\} \\ &= a (3 \sin \theta + \sin 3\theta) = a (6 \sin \theta - 4 \sin^3 \theta). \\ \therefore 2(x + y) &= a (\cos \theta + \sin \theta) \{6 - 4(\cos^2 \theta - \cos \theta \sin \theta + \sin^2 \theta)\} \\ &= 2a (\cos \theta + \sin \theta) (1 + \sin 2\theta); \\ \therefore x + y &= a (\cos \theta + \sin \theta)^3. \end{aligned}$$

Similarly,

$$\begin{aligned} x - y &= a (\cos \theta - \sin \theta)^3; \\ \therefore (x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} &= 2a^{\frac{2}{3}}. \end{aligned}$$

21. From the first equation,

$$(x \sin \theta - y \cos \theta)^2 = (x^2 + y^2) (\sin^2 \theta + \cos^2 \theta);$$

whence

$$x \cos \theta + y \sin \theta = 0;$$

$$\therefore \frac{\cos \theta}{y} = \frac{\sin \theta}{-x} = \frac{1}{\sqrt{x^2 + y^2}}.$$

By substituting in the second equation, we have

$$\frac{1}{x^2 + y^2} \left(\frac{y^2}{a^2} + \frac{x^2}{b^2} \right) = \frac{1}{x^2 + y^2};$$

or

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

22. From the first equation, we have

$$bx \cos \theta + ay \sin \theta = ab \sqrt{\sin^2 \theta + \cos^2 \theta};$$

$$\therefore b^2 (x^2 - a^2) \cos^2 \theta + 2abxy \cos \theta \sin \theta + a^2 (y^2 - b^2) \sin^2 \theta = 0.$$

From the second equation, we have

$$(y^2 - b^2) \cos^2 \theta - 2xy \cos \theta \sin \theta + (x^2 - a^2) \sin^2 \theta = 0.$$

Multiplying this equation by ab and adding to the preceding equation,

$$\{b^2 (x^2 - a^2) + ab (y^2 - b^2)\} \cos^2 \theta + \{a^2 (y^2 - b^2) + ab (x^2 - a^2)\} \sin^2 \theta = 0;$$

$$\therefore \{bx^2 + ay^2 - ab (a + b)\} (b \cos^2 \theta + a \sin^2 \theta) = 0.$$

$$\therefore bx^2 + ay^2 - ab (a + b) = 0,$$

or

$$b \cos^2 \theta + a \sin^2 \theta = 0.$$

From the last result,
$$\frac{\cos^2 \theta}{a} = \frac{\sin^2 \theta}{-b};$$

but
$$\{(y^2 - b^2) \cos^2 \theta + (x^2 - a^2) \sin^2 \theta\}^2 = 4x^2 y^2 \cos^2 \theta \sin^2 \theta;$$

$$\therefore \{a (y^2 - b^2) - b (x^2 - a^2)\}^2 = -4abx^2 y^2.$$

23. We have $4 \cos (\alpha - 3\theta) = m (3 \cos \theta + \cos 3\theta),$

and $4 \sin (\alpha - 3\theta) = m (3 \sin \theta - \sin 3\theta);$

whence by squaring and adding, we have

$$16 = m^2 (10 + 6 \cos 4\theta).$$

Again, by multiplying the first equation by $\cos 3\theta$ and the second by $\sin 3\theta$ and subtracting, we obtain

$$4 \cos \alpha = m (3 \cos 4\theta + 1);$$

$$\therefore 16 = 10m^2 + 2m (4 \cos \alpha - m),$$

or

$$2 = m^2 + m \cos \alpha.$$

24. We have
$$\frac{x}{y} = \frac{\tan \theta + \tan \phi}{\cot \theta + \cot \phi} = \tan \theta \tan \phi,$$

But
$$\tan \alpha = \tan (\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi};$$

$$\therefore \tan \alpha = \frac{xy}{y - x}.$$

25. We have
$$a^2 + b^2 = 2 + 2 \cos (\theta - \phi);$$

$$\therefore a^2 + b^2 = 2 + 2 \cos \alpha.$$

26. We have
$$a \sin^2 \theta + b \cos^2 \theta = 1 = \sin^2 \theta + \cos^2 \theta;$$

$$\therefore (a - 1) \tan^2 \theta = 1 - b.$$

Again,

$$a \cos^2 \phi + b \sin^2 \phi = 1 = \cos^2 \phi + \sin^2 \phi;$$

$$\therefore (b-1) \tan^2 \phi = 1-a.$$

But

$$a^2 \tan^2 \theta = b^2 \tan^2 \phi;$$

$$\therefore \frac{a^2(1-b)}{a-1} = \frac{b^2(1-a)}{b-1};$$

$$\therefore a(b-1) = \pm b(a-1).$$

Rejecting the upper sign, we have $a+b=2ab$.

27. From the first two equations, we have

$$\frac{\frac{x}{a}}{\cos \frac{\theta+\phi}{2}} = \frac{\frac{y}{b}}{\sin \frac{\theta+\phi}{2}} = \frac{1}{\cos \frac{\theta-\phi}{2}};$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2 \frac{\theta-\phi}{2} = \sec^2 \frac{\alpha}{2}.$$

28. We have

$$\frac{a}{b} = \frac{\tan \theta + \tan \phi}{\cot \theta + \cot \phi} = \tan \theta \tan \phi.$$

But

$$\tan \alpha = \tan (\theta - \phi) = \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi};$$

$$\therefore (a+b) \tan \alpha = b (\tan \theta - \tan \phi);$$

$$\begin{aligned} \therefore (a+b)^2 \tan^2 \alpha &= b^2 \{(\tan \theta + \tan \phi)^2 - 4 \tan \theta \tan \phi\} \\ &= b^2 a^2 - 4ab. \end{aligned}$$

29. We have

$$\frac{a \cos^2 \theta + b \sin^2 \theta}{a \sin^2 \theta + b \cos^2 \theta} = \frac{m \cos^2 \phi}{n \sin^2 \phi};$$

$$\therefore \frac{a+b \tan^2 \theta}{a \tan^2 \theta + b} = \frac{m}{n \tan^2 \phi} = \frac{1}{\tan^2 \theta};$$

$$\therefore b \tan^4 \theta = b, \text{ or } \tan^2 \theta = \pm 1.$$

$$\therefore n \tan^2 \phi = \pm m.$$

By adding together the first two equations, we obtain

$$a+b = m \cos^2 \phi + n \sin^2 \phi.$$

If $n \tan^2 \phi = m$, then

$$\frac{\cos^2 \phi}{n} = \frac{\sin^2 \phi}{m} = \frac{1}{m+n};$$

$$\therefore a+b = \frac{2mn}{m+n}.$$

If $n \tan^2 \phi = -m$, we obtain $a+b=0$.

30. From the second and third equations, we have by addition

$$2 \cos \phi (x \cos \theta + y \sin \theta) = 6a;$$

$$\therefore 2a \sqrt{3} \cos \phi = 3a, \text{ whence } \cos \phi = \frac{\sqrt{3}}{2}.$$

From the second and third equations, we have by subtraction

$$2 \sin \phi (-x \sin \theta + y \cos \theta) = 2a;$$

$$\therefore -x \sin \theta + y \cos \theta = \pm 2a.$$

But

$$x \cos \theta + y \sin \theta = 2a \sqrt{3};$$

$$\therefore x^2 + y^2 = 16a^2.$$

31. We have $c \sin \theta = a (\sin \theta \cos \phi + \cos \theta \sin \phi);$

$$\therefore (c - a \cos \phi) \sin \theta = a \sin \phi \cos \theta = b \sin \theta \cdot \cos \theta;$$

$$\therefore c - a \cos \phi = b \cos \theta = b (2m + \cos \phi);$$

$$\therefore \cos \phi = \frac{c - 2bm}{a + b}.$$

And

$$\cos \theta = 2m + \cos \phi = \frac{c + 2am}{a + b}.$$

But

$$a \sin \phi = b \sin \theta;$$

$$\therefore a^2 - a^2 \cos^2 \phi = b^2 - b^2 \cos^2 \theta;$$

$$\therefore a^2 (a + b)^2 - a^2 (c - 2bm)^2 = b^2 (a + b)^2 - b^2 (c + 2am)^2;$$

$$\therefore (a^2 - b^2) (a + b)^2 = c^2 (a^2 - b^2) - 4abcm (a + b);$$

$$\therefore 4abcm = (a - b) \{c^2 - (a + b)^2\}.$$

EXAMPLES. XXV. c. PAGE 334.

1. By putting $x = y \cos \theta$, the given equation becomes

$$\cos^3 \theta - \frac{3}{y^2} \cos \theta - \frac{1}{y^3} = 0.$$

But

$$\cos^3 \theta - \frac{3}{4} \cos \theta - \frac{\cos 3\theta}{4} = 0;$$

$$\therefore \frac{3}{y^2} = \frac{3}{4}; \text{ whence } y = 2.$$

Also

$$\frac{\cos 3\theta}{4} = \frac{1}{y^3} = \frac{1}{8}; \text{ whence } \cos 3\theta = \frac{1}{2};$$

$$\therefore 3\theta = n \cdot 360^\circ \pm 60^\circ;$$

$$\therefore \theta = 20^\circ, 100^\circ, \text{ or } 140^\circ.$$

But $x = y \cos \theta = 2 \cos \theta$; and therefore the roots are

$$2 \cos 20^\circ, \quad -2 \cos 40^\circ, \quad -2 \cos 80^\circ.$$

2. By putting $x = y \sin \theta$, the given equation becomes

$$\sin^3 \theta - \frac{3}{y^2} \sin \theta + \frac{1}{y^3} = 0.$$

But

$$\sin^3 \theta - \frac{3}{4} \sin \theta + \frac{\sin 3\theta}{4} = 0;$$

$$\therefore y^2 = 4; \text{ whence } y = 2.$$

Also

$$\frac{\sin 3\theta}{4} = \frac{1}{y^3} = \frac{1}{8}; \text{ whence } \sin 3\theta = \frac{1}{2}.$$

Hence as in Art. 328, the roots of the equation are

$$2 \sin 10^\circ, \quad 2 \sin 50^\circ, \quad -2 \sin 70^\circ.$$

3. Put $x = y \cos \theta$, then $\cos^3 \theta - \frac{3}{y^2} \cos \theta - \frac{\sqrt{3}}{y^3} = 0.$

But $\cos^3 \theta - \frac{3}{4} \cos \theta - \frac{\cos 3\theta}{4} = 0;$

$$\therefore y^2 = 4; \text{ whence } y = 2.$$

Also $\frac{\cos 3\theta}{4} = \frac{\sqrt{3}}{y^3} = \frac{\sqrt{3}}{8}; \text{ whence } \cos 3\theta = \frac{\sqrt{3}}{2};$

$$\therefore 3\theta = n \cdot 360^\circ \pm 30^\circ; \therefore \theta = 10^\circ, 110^\circ, \text{ or } 130^\circ.$$

But $x = y \cos \theta = 2 \cos \theta$, and therefore the roots are

$$2 \cos 10^\circ, \quad -2 \cos 50^\circ, \quad -2 \cos 70^\circ.$$

4. Put $x = y \sin \theta$; then $\sin^3 \theta - \frac{3}{4y^2} \sin \theta + \frac{\sqrt{2}}{8y^3} = 0.$

But $\sin^3 \theta - \frac{3}{4} \sin \theta + \frac{\sin 3\theta}{4} = 0;$

$$\therefore 4y^2 = 4; \text{ whence } y = 1.$$

Also since $\frac{\sin 3\theta}{4} = \frac{\sqrt{2}}{8y^3} = \frac{\sqrt{2}}{8}; \text{ whence } \sin 3\theta = \frac{1}{\sqrt{2}};$

$$\therefore 3\theta = n \cdot 180^\circ + (-1)^n 45^\circ;$$

$$\therefore \theta = 15^\circ, 45^\circ, 135^\circ, 165^\circ, 255^\circ, \dots$$

$$\therefore \sin \theta = \sin 15^\circ, \sin 45^\circ, \sin 255^\circ.$$

But $x = y \sin \theta = \sin \theta$, and therefore the roots are

$$\sin 15^\circ, \quad \sin 45^\circ, \quad -\sin 75^\circ.$$

5. Put $x = y \sin \theta$; then $\sin^3 \theta - \frac{3}{4a^2y^2} \sin \theta + \frac{\sin 3\theta}{4a^3y^3} = 0.$

But $\sin^3 \theta - \frac{3}{4} \sin \theta + \frac{\sin 3\theta}{4} = 0;$

$$\therefore 4a^2y^2 = 4; \text{ whence } y = \frac{1}{a}.$$

Also
$$\frac{\sin 3\theta}{4} = \frac{\sin 3A}{4a^3y^3} = \frac{\sin 3A}{4};$$

$$\therefore 3\theta = n \cdot 180^\circ + (-1)^n 3A;$$

$$\therefore \theta = A, \quad 60^\circ - A, \quad 120^\circ + A, \quad 180^\circ - A, \quad 240^\circ + A, \dots$$

$$\therefore \sin \theta = \sin A, \quad \sin (60^\circ - A), \quad \sin (240^\circ + A).$$

But $x = y \sin \theta = \frac{1}{a} \sin \theta$, and therefore the roots are

$$\frac{1}{a} \sin A, \quad \frac{1}{a} \sin (60^\circ - A), \quad -\frac{1}{a} \sin (60^\circ + A).$$

6. Put $x = y \cos \theta$; then $\cos^3 \theta - \frac{3a^2}{y^2} \cos \theta - \frac{2a^3 \cos 3A}{y^3} = 0.$

But
$$\cos^3 \theta - \frac{3}{4} \cos \theta - \frac{\cos 3\theta}{4} = 0;$$

$$\therefore \frac{3a^2}{y^2} = \frac{3}{4}; \text{ whence } y = 2a.$$

Also
$$\frac{\cos 3\theta}{4} = \frac{2a^3 \cos 3A}{y^3} = \frac{\cos 3A}{4};$$

$$\therefore 3\theta = n \cdot 360^\circ \pm 3A;$$

$$\therefore \theta = A, \quad 120^\circ \pm A.$$

But $x = y \cos \theta = 2a \cos \theta$, and therefore the roots are

$$2a \cos A, \quad 2a \cos (120^\circ \pm A).$$

7. (1) From the theory of quadratic equations, we have

$$\sin \alpha + \sin \beta = -\frac{b}{a} \dots \dots \dots (1).$$

By supposition, $\sin \alpha + 2 \sin \beta = 1,$

$$\therefore \sin \beta = 1 + \frac{b}{a}.$$

But $a \sin^2 \beta + b \sin \beta + c = 0 \dots \dots \dots (2),$

$$\therefore (a+b)^2 + b(a+b) + ac = 0.$$

(2) Substituting from the equation $c \sin \alpha = a \sin \beta$ in (1), we have

$$a \sin \beta + c \sin \beta = -\frac{bc}{a};$$

$$\therefore a(a+c) \sin \beta = -bc.$$

But $\sin \alpha \sin \beta = \frac{c}{a};$

$$\therefore a^2 \sin^2 \beta = c^2; \text{ whence } a+c = \pm b.$$

8. We have $\tan \alpha + \tan \beta = \frac{b}{a}.$

Also, by hypothesis, $a \tan \alpha + b \tan \beta = 2b;$
whence $(b-a) \tan \beta = b.$

But $a \tan^2 \beta - b \tan \beta + c = 0;$
 $\therefore ab^2 - b^2(b-a) + c(b-a)^2 = 0.$

9. We have $\tan \alpha + \tan \beta + \tan \gamma = 0,$
and $\tan \alpha \tan \beta + \tan \alpha \tan \gamma + \tan \beta \tan \gamma = \frac{2a-x}{a};$

$$\therefore \tan \alpha \tan \beta - \tan^2 \gamma = \frac{2a-x}{a}.$$

Now $\tan^2 \alpha + \tan^2 \beta = (\tan \alpha + \tan \beta)^2 - 2 \tan \alpha \tan \beta$
 $= \tan^2 \gamma - 2 \tan^2 \gamma - \frac{2(2a-x)}{a};$
 $\therefore a(\tan^2 \alpha + \tan^2 \beta) = -a \tan^2 \gamma - 4a + 2x.$

But, by hypothesis, $a(\tan^2 \alpha + \tan^2 \beta) = 2x - 5a;$
 $\therefore 2x - 5a = -a \tan^2 \gamma - 4a + 2x;$
 $\therefore \tan^2 \gamma = 1.$

But $a \tan^3 \gamma + (2a-x) \tan \gamma + y = 0;$
 $\therefore a \tan \gamma + (2a-x) \tan \gamma + y = 0;$
 $\therefore (3a-x) \tan \gamma + y = 0;$
 $\therefore 3a-x = \pm y.$

10. We have $\cos \alpha \cos \beta + \cos \alpha \cos \gamma + \cos \beta \cos \gamma = b.$

Also, by supposition, $\cos \alpha \cos \beta + \cos \alpha \cos \gamma = 2b;$
 $\therefore \cos \beta \cos \gamma = -b.$

Again, $\cos \alpha \cos \beta \cos \gamma = -c;$
 $\therefore b \cos \alpha = c.$

Now $\cos^3 \alpha + a \cos^2 \alpha + b \cos \alpha + c = 0;$
 $\therefore c^3 + abc^2 + b^3 c + b^3 c = 0.$

11. As on page 329 we may shew that

$$\cos \alpha, \quad \cos \left(\frac{2\pi}{3} + \alpha \right), \quad \cos \left(\frac{2\pi}{3} - \alpha \right)$$

are the roots of the cubic

$$\cos^3 \theta - \frac{3}{4} \cos \theta - \frac{\cos 3\alpha}{4} = 0.$$

Write $\cos \theta = \frac{1}{\sec \theta}$; then $\frac{\cos 3\alpha}{4} \sec^3 \theta + \frac{3}{4} \sec^2 \theta - 1 = 0$;

$$\therefore \sec \alpha + \sec \left(\frac{2\pi}{3} + \alpha \right) + \sec \left(\frac{2\pi}{3} - \alpha \right) = -\frac{3}{4} \div \frac{\cos 3\alpha}{4}.$$

12. The values of $\sin \theta$ found from the equation $\sin 3\theta = \sin 3\alpha$ are

$$\sin \alpha, \quad \sin \left(\frac{2\pi}{3} + \alpha \right), \quad \sin \left(\frac{4\pi}{3} + \alpha \right);$$

that is, these three quantities are the roots of the cubic equation

$$\sin^3 \theta - \frac{3}{4} \sin \theta + \frac{\sin 3\alpha}{4} = 0 \dots\dots\dots(1).$$

$\therefore S_1 = \text{Sum of the roots} = 0$;

and $S_2 = \text{Sum of products of the roots taken two at a time} = -\frac{3}{4}$;

$$\begin{aligned} \therefore \sin^2 \alpha + \sin^2 \left(\frac{2\pi}{3} + \alpha \right) + \sin^2 \left(\frac{4\pi}{3} + \alpha \right) &= S_1^2 - 2S_2 \\ &= 0 - 2 \left(-\frac{3}{4} \right) = \frac{3}{2}. \end{aligned}$$

13. In equation (1) of Example 12, put $\sin \theta = \frac{1}{\operatorname{cosec} \theta}$; then

$$\frac{\sin 3\alpha}{4} \operatorname{cosec}^3 \theta - \frac{3}{4} \operatorname{cosec}^2 \theta + 1 = 0.$$

$$\therefore \operatorname{cosec} \alpha + \operatorname{cosec} \left(\frac{2\pi}{3} + \alpha \right) + \operatorname{cosec} \left(\frac{4\pi}{3} + \alpha \right) = \frac{3}{4} \div \frac{\sin 3\alpha}{4}.$$

14. On p. 330 we have shewn that $\sin^2 \frac{\pi}{5}$ and $\sin^2 \frac{2\pi}{5}$ are the roots of the quadratic equation $16x^2 - 20x + 5 = 0$.

Put $x = \frac{1}{y}$; then $5y^2 - 20y + 16 = 0$, is an equation whose roots are

$$\operatorname{cosec}^2 \frac{\pi}{5} \quad \text{and} \quad \operatorname{cosec}^2 \frac{2\pi}{5};$$

$$\therefore \operatorname{cosec}^2 \frac{\pi}{5} + \operatorname{cosec}^2 \frac{2\pi}{5} = \frac{20}{5} = 4.$$

15. If $\cos 3\theta = \cos 2\theta$,
 then $4 \cos^3 \theta - 2 \cos^2 \theta - 3 \cos \theta + 1 = 0$,
 or $(\cos \theta - 1)(4 \cos^2 \theta + 2 \cos \theta - 1) = 0$.

But the roots of $\cos 3\theta = \cos 2\theta$ considered as a cubic in $\cos \theta$ are

$$1, \quad \cos \frac{2\pi}{5}, \quad \cos \frac{4\pi}{5}. \quad [\text{Art. 331.}]$$

$\therefore \cos \frac{2\pi}{5}$ and $\cos \frac{4\pi}{5}$ are the roots of $4 \cos^2 \theta + 2 \cos \theta - 1 = 0$;

$$\therefore \cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{2}{4} = -\frac{1}{2}.$$

and $\cos \frac{2\pi}{5} \cos \frac{4\pi}{5} = -\frac{1}{4}.$

16. (1) Let $7\theta = n\pi$, where n is any *odd* integer;

then $4\theta = n\pi - 3\theta$, and $\cos 4\theta = -\cos 3\theta$.

$$\therefore 8 \cos^4 \theta - 8 \cos^2 \theta + 1 = -4 \cos^3 \theta + 3 \cos \theta;$$

$$\therefore (\cos \theta + 1)(8 \cos^3 \theta - 4 \cos^2 \theta - 4 \cos \theta + 1) = 0 \dots \dots \dots (1).$$

The roots of the equation $\cos 4\theta = -\cos 3\theta$ considered as a biquadratic in $\cos \theta$ are $\cos \frac{\pi}{7}$, $\cos \frac{3\pi}{7}$, $\cos \frac{5\pi}{7}$, $\cos \frac{7\pi}{7}$, the last of which corresponds to the factor $\cos \theta + 1$ in equation (1). Hence the equation whose roots are $\cos \frac{\pi}{7}$, $\cos \frac{3\pi}{7}$, $\cos \frac{5\pi}{7}$ is

$$8 \cos^3 \theta - 4 \cos^2 \theta - 4 \cos \theta + 1 = 0.$$

(2) Let y denote any one of the quantities

$$\sin^2 \frac{\pi}{14}, \quad \sin^2 \frac{3\pi}{14}, \quad \sin^2 \frac{5\pi}{14}.$$

then $2y = 1 - x$, where x denotes one of the quantities

$$\cos \frac{\pi}{7}, \quad \cos \frac{3\pi}{7}, \quad \cos \frac{5\pi}{7}.$$

But we have seen in the first part of this question that these quantities are the roots of the cubic

$$8x^3 - 4x^2 - 4x + 1 = 0.$$

Substituting $x = 1 - 2y$, we have

$$8(1 - 2y)^3 - 4(1 - 2y)^2 - 4(1 - 2y) + 1 = 0,$$

or $64y^3 - 80y^2 + 24y - 1 = 0.$

17. Let y denote one of the quantities $\sin^2 \frac{\pi}{7}$, $\sin^2 \frac{2\pi}{7}$, $\sin^2 \frac{3\pi}{7}$, then $2y = 1 - x$; where x denotes one of the quantities

$$\cos \frac{2\pi}{7}, \cos \frac{4\pi}{7}, \cos \frac{6\pi}{7}.$$

But in Art. 331, we have shewn that these quantities are the roots of the cubic

$$8x^3 + 4x^2 - 4x - 1 = 0.$$

Substituting $x = 1 - 2y$, we have

$$8(1 - 2y)^3 + 4(1 - 2y)^2 - 4(1 - 2y) - 1 = 0,$$

or

$$64y^3 - 112y^2 + 56y - 7 = 0,$$

the roots of which are $\sin^2 \frac{\pi}{7}$, $\sin^2 \frac{2\pi}{7}$, $\sin^2 \frac{3\pi}{7}$.

$$\therefore \sin^4 \frac{\pi}{7} + \sin^4 \frac{2\pi}{7} + \sin^4 \frac{3\pi}{7} = \left(\frac{112}{64}\right)^2 - 2\left(\frac{56}{64}\right) = \frac{21}{16}.$$

Put $y = \frac{1}{z}$; then $7z^3 - 56z^2 + 112z - 64 = 0$ is an equation whose roots are

$$\operatorname{cosec}^2 \frac{\pi}{7}, \operatorname{cosec}^2 \frac{2\pi}{7}, \operatorname{cosec}^2 \frac{3\pi}{7}.$$

$$\therefore \operatorname{cosec}^4 \frac{\pi}{7} + \operatorname{cosec}^4 \frac{2\pi}{7} + \operatorname{cosec}^4 \frac{3\pi}{7} = \left(\frac{56}{7}\right)^2 - 2\left(\frac{112}{7}\right) = 32.$$

18. (1) As in Art. 331 the required equation is $\cos 5\theta = \cos 4\theta$.

Expressing $\cos 5\theta$ and $\cos 4\theta$ in terms of $\cos \theta$, we have

$$16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1. \quad [\text{See Art. 332.}]$$

By transposition and removal of the factor $\cos \theta - 1$ we obtain

$$16 \cos^4 \theta + 8 \cos^3 \theta - 12 \cos^2 \theta - 4 \cos \theta + 1 = 0,$$

which is the equation required.

(2) Put $\cos \theta = x$, then the above equation becomes

$$16x^4 + 8x^3 - 12x^2 - 4x + 1 = 0.$$

Now $\cos \frac{\pi}{9} = -\cos \frac{8\pi}{9}$, $\cos \frac{3\pi}{9} = -\cos \frac{6\pi}{9}$, ...;

hence by writing $x = -y$, we see that

$$\cos \frac{\pi}{9}, \cos \frac{3\pi}{9}, \cos \frac{5\pi}{9}, \cos \frac{7\pi}{9}$$

are the roots of the equation $16y^4 - 8y^3 - 12y^2 + 4y + 1 = 0$.

The same result may be arrived at by putting $9\theta = n\pi$, where n is any odd integer. For $5\theta = n\pi - 4\theta$, so that $\cos 5\theta = -\cos 4\theta$. [See Example 16 (1).]

19. Let y denote one of the quantities

$$\cos^2 \frac{\pi}{9}, \cos^2 \frac{2\pi}{9}, \cos^2 \frac{3\pi}{9}, \cos^2 \frac{4\pi}{9} \dots\dots\dots(1),$$

then $2y=1+x$, where x denotes one of the quantities

$$\cos \frac{2\pi}{9}, \cos \frac{4\pi}{9}, \cos \frac{6\pi}{9}, \cos \frac{8\pi}{9}.$$

But these quantities are by the last example the roots of the equation

$$16x^4+8x^3-12x^2-4x+1=0;$$

hence the given quantities are the roots of

$$16(2y-1)^4+8(2y-1)^3-12(2y-1)^2-4(2y-1)+1=0,$$

or
$$256y^4-448y^3+240y^2-40y+1=0.$$

$$\begin{aligned} \therefore \cos^4 \frac{\pi}{9} + \cos^4 \frac{2\pi}{9} + \cos^4 \frac{3\pi}{9} + \cos^4 \frac{4\pi}{9} &= \left(\frac{448}{256}\right)^2 - 2\left(\frac{240}{256}\right) \\ &= \frac{49}{16} - \frac{30}{16} = \frac{19}{16}. \end{aligned}$$

If we put $y=\frac{1}{z}$, we obtain

$$z^4-40z^3+240z^2-448z+256=0,$$

the roots of which are

$$\sec^2 \frac{\pi}{9}, \sec^2 \frac{2\pi}{9}, \sec^2 \frac{3\pi}{9}, \sec^2 \frac{4\pi}{9}.$$

$$\therefore \sec^4 \frac{\pi}{9} + \sec^4 \frac{2\pi}{9} + \sec^4 \frac{3\pi}{9} + \sec^4 \frac{4\pi}{9} = (40)^2 - 2 \times 240 = 1120.$$

20. As on p. 332, we may shew that the given quantities are the roots of the equation $\tan 5\theta = -\tan 4\theta$, considered as an equation in $\tan \theta$.

Put $\tan \theta = t$, then $\tan 5\theta = \tan (3\theta + 2\theta)$;

$$\therefore \tan 5\theta = \frac{\frac{3t-t^3}{1-3t^2} + \frac{2t}{1-t^2}}{1 - \frac{2t(3t-t^3)}{(1-3t^2)(1-t^2)}} = \frac{5t-10t^3+t^5}{1-10t^2+5t^4}.$$

Hence the equation $\tan 5\theta = -\tan 4\theta$ becomes

$$\frac{t^5-10t^3+5t}{5t^4-10t^2+1} - \frac{4t^3-4t}{t^4-6t^2+1} = 0;$$

$$\begin{aligned} \therefore (t^8-16t^6+66t^4-40t^2+5) - (20t^6-60t^4+4t^2-4) &= 0; \\ \therefore t^8-36t^6+126t^4-84t^2+9 &= 0, \end{aligned}$$

which is a biquadratic in t^2 having for roots

$$\tan^2 \frac{\pi}{9}, \tan^2 \frac{2\pi}{9}, \tan^2 \frac{3\pi}{9}, \tan^2 \frac{4\pi}{9}.$$

Put $t^2 = \frac{1}{x}$, then

$$9x^4 - 84x^3 + 126x^2 - 36 + 1 = 0,$$

an equation whose roots are

$$\cot^2 \frac{\pi}{9}, \cot^2 \frac{2\pi}{9}, \cot^2 \frac{3\pi}{9}, \cot^2 \frac{4\pi}{9}.$$

$$\text{But } \cot^2 \frac{3\pi}{9} = \cot^2 \frac{\pi}{3} = \frac{1}{3}.$$

$$\therefore \cot^2 \frac{\pi}{9} + \cot^2 \frac{2\pi}{9} + \cot^2 \frac{4\pi}{9} + \frac{1}{3} = \frac{84}{9} = 9\frac{1}{3}.$$

$$\therefore \cot^2 \frac{\pi}{9} + \cot^2 \frac{2\pi}{9} + \cot^2 \frac{4\pi}{9} = 9.$$

21. (1) As in Ex. 1, p. 332, we can shew that $t^6 - 21t^4 + 35t^2 - 7 = 0$ is an equation whose roots are

$$\tan^2 \frac{\pi}{7}, \tan^2 \frac{2\pi}{7}, \tan^2 \frac{3\pi}{7};$$

$\therefore$ writing $\frac{1}{c}$ for t , we obtain

$$7c^6 - 35c^4 + 21c^2 - 1 = 0,$$

which is a cubic in c^2 whose roots are

$$\cot^2 \frac{\pi}{7}, \cot^2 \frac{2\pi}{7}, \cot^2 \frac{3\pi}{7}.$$

$$\therefore \cot^2 \frac{\pi}{7} + \cot^2 \frac{2\pi}{7} + \cot^2 \frac{3\pi}{7} = 5;$$

$$\therefore \operatorname{cosec}^2 \frac{\pi}{7} + \operatorname{cosec}^2 \frac{2\pi}{7} + \operatorname{cosec}^2 \frac{3\pi}{7} = 8.$$

$$(2) \text{ Here } \cos \frac{\pi}{11} = -\cos \frac{10\pi}{11}, \cos \frac{3\pi}{11} = -\cos \frac{8\pi}{11}, \cos \frac{5\pi}{11} = -\cos \frac{6\pi}{11}.$$

$$\begin{aligned} \therefore \cos \frac{\pi}{11} \cos \frac{2\pi}{11} \cos \frac{3\pi}{11} \cos \frac{4\pi}{11} \cos \frac{5\pi}{11} \\ = -\cos \frac{2\pi}{11} \cos \frac{4\pi}{11} \cos \frac{6\pi}{11} \cos \frac{8\pi}{11} \cos \frac{10\pi}{11}. \end{aligned}$$

Now $\cos \frac{2\pi}{11}, \cos \frac{4\pi}{11}, \dots, \cos \frac{10\pi}{11}$ are the roots of the equation

$$\cos 6\theta = \cos 5\theta.$$

[See Art. 331.]

Using the expressions for $\cos 6\theta$ and $\cos 5\theta$ given in Art. 332 and putting x for $\cos \theta$, we have

$$32x^6 - 48x^4 + 18x^2 - 1 = 16x^5 - 20x^3 + 5x,$$

or
$$32x^6 - 16x^5 - 48x^4 + 20x^3 + 18x^2 - 5x - 1 = 0.$$

Removing the factor $x - 1$, which corresponds to $\cos \theta = 1$, we have

$$32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1 = 0.$$

The product of the roots is $-\frac{1}{32}$, and therefore the value of the required expression is $\frac{1}{32}$.

MISCELLANEOUS EXAMPLES. I. PAGE 336.

1. By transposition, we have

$$a \left(\tan a - \tan \frac{a+\beta}{2} \right) = b \left(\tan \frac{a+\beta}{2} - \tan \beta \right);$$

$$\therefore \frac{a \sin \left(a - \frac{a+\beta}{2} \right)}{\cos a \cos \frac{a+\beta}{2}} = \frac{b \sin \left(\frac{a+\beta}{2} - \beta \right)}{\cos \beta \cos \frac{a+\beta}{2}};$$

$$\therefore \frac{a}{\cos a} = \frac{b}{\cos \beta}.$$

2. We have
$$\frac{a+b}{a} \sin^4 a + \frac{a+b}{b} \cos^4 a = 1;$$

$$\therefore \left(1 + \frac{b}{a} \right) \sin^4 a + \left(1 + \frac{a}{b} \right) \cos^4 a = \sin^4 a + 2 \sin^2 a \cos^2 a + \cos^4 a;$$

$$\therefore \frac{b}{a} \sin^4 a - 2 \sin^2 a \cos^2 a + \frac{a}{b} \cos^4 a = 0;$$

$$\therefore \frac{\sin^4 a}{a^2} - \frac{2 \sin^2 a \cos^2 a}{ab} + \frac{\cos^4 a}{b^2} = 0;$$

$$\therefore \frac{\sin^2 a}{a} = \frac{\cos^2 a}{b} = \frac{1}{a+b}.$$

$$\therefore \frac{\sin^8 a}{a^3} + \frac{\cos^8 a}{b^3} = \frac{a}{(a+b)^4} + \frac{b}{(a+b)^4} = \frac{1}{(a+b)^3}.$$

$$\begin{aligned}
6. \text{ (i) } & 4\Sigma \sin(\beta - \gamma) \cos(\alpha - \beta) \cos(\alpha - \gamma) \\
& = 2\Sigma \sin(\beta - \gamma) \{\cos(2\alpha - \beta - \gamma) + \cos(\beta - \gamma)\} \\
& = 2\Sigma \sin(\beta - \gamma) \cos\{2\alpha - (\beta + \gamma)\} + 2\Sigma \sin(\beta - \gamma) \cos(\beta - \gamma) \\
& = \Sigma \{\sin(2\alpha - 2\gamma) + \sin(2\beta - 2\alpha)\} + \Sigma \sin 2(\beta - \gamma) \\
& = -2\Sigma \sin 2(\beta - \gamma) + \Sigma \sin 2(\beta - \gamma) \\
& = -\Sigma \sin 2(\beta - \gamma) \\
& = 4\Pi \sin(\beta - \gamma). \quad [\text{Ex. 2, page 304.}]
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } & 4\Sigma \sin \alpha \sin(\beta - \gamma) \cos(\beta + \gamma - \alpha) \\
& = 2\Sigma \sin \alpha \{\sin(2\beta - \alpha) + \sin(\alpha - 2\gamma)\} \\
& = 2\Sigma \sin \alpha \sin(2\beta - \alpha) + 2\Sigma \sin \alpha \sin(\alpha - 2\gamma) \\
& = \Sigma \{\cos 2(\alpha - \beta) - \cos 2\beta\} + \Sigma \{\cos 2\gamma - \cos 2(\alpha - \gamma)\} \\
& = \Sigma \{\cos 2(\alpha - \beta) - \cos 2(\gamma - \alpha)\} + \Sigma (\cos 2\gamma - \cos 2\beta) \\
& = 0.
\end{aligned}$$

$$\begin{aligned}
\text{(iii) } & 4\Sigma \sin \alpha \sin(\beta - \gamma) \sin(\beta + \gamma - \alpha) \\
& = 2\Sigma \sin \alpha \{\cos(\alpha - 2\gamma) - \cos(2\beta - \alpha)\} \\
& = \Sigma \{\sin(2\alpha - 2\gamma) + \sin 2\gamma\} - \Sigma \{\sin 2\beta + \sin(2\alpha - 2\beta)\} \\
& = -\Sigma \sin 2(\gamma - \alpha) + \Sigma (\sin 2\gamma - \sin 2\beta) - \Sigma \sin 2(\alpha - \beta) \\
& = -2\Sigma \sin 2(\alpha - \beta) \\
& = 8\Pi \sin(\alpha - \beta). \quad [\text{Ex. 2, page 304.}]
\end{aligned}$$

7. (1) Let x, y, z denote the lengths of PA, PB, PC respectively; and let the areas of the triangles PBC, PCA, PAB be denoted by $\delta_1, \delta_2, \delta_3$ respectively.

Then in the triangle PBC

$$\begin{aligned}
\cot \omega &= \frac{\cos \omega}{\sin \omega} = \frac{a^2 + y^2 - z^2}{2ay \sin \omega} = \frac{a^2 + y^2 - z^2}{4\delta_1}; \\
\therefore \cot \omega &= \frac{a^2 + y^2 - z^2}{4\delta_1} = \frac{b^2 + z^2 - x^2}{4\delta_2} = \frac{c^2 + x^2 - y^2}{4\delta_3} \\
&= \frac{a^2 + b^2 + c^2}{4(\delta_1 + \delta_2 + \delta_3)} = \frac{a^2 + b^2 + c^2}{4\Delta} \\
&= \cot A + \cot B + \cot C. \quad [\text{XVIII. a., Ex. 34.}]
\end{aligned}$$

(2) By squaring the result just obtained, we have

$$\begin{aligned}
\cot^2 \omega &= \cot^2 A + \cot^2 B + \cot^2 C + 2\Sigma \cot B \cot C \\
&= \cot^2 A + \cot^2 B + \cot^2 C + 2,
\end{aligned}$$

since

$$\Sigma \cot B \cot C = 1.$$

$$\therefore 1 + \cot^2 \omega = (1 + \cot^2 A) + (1 + \cot^2 B) + (1 + \cot^2 C);$$

$$\therefore \operatorname{cosec}^2 \omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C.$$

8. On page 195 of the *Elementary Trigonometry*, suppose that $ABFD$ is a vertical plane running N. and S., and that CG is drawn in a S.E. direction. Also suppose that $AB = a$, and $AC = 169a$;

then $BC^2 = (169a)^2 - a^2 = 28560a^2$;

$$\therefore CG^2 = 2BC^2 = 57120a^2;$$

$$\therefore CH^2 = 57121a^2,$$

$$\therefore CH = 239a.$$

9. Let ABC be a horizontal section of the two walls, and let ϕ be the inclination of the wall AB to the meridian, so that $\gamma - \phi$ is the inclination of BC to the meridian.

The length of the shadow of the wall AB measured along the meridian is $a \cot \theta$; hence the breadth of the shadow (which is measured at right angles to AB) is $a \cot \theta \sin \phi$,

$$\therefore b = a \cot \theta \sin \phi.$$

Similarly,

$$c = a \cot \theta \sin (\gamma - \phi);$$

$$\therefore c = a \cot \theta (\sin \gamma \cos \phi - \cos \gamma \sin \phi)$$

$$= a \cot \theta \sin \gamma \cos \phi - b \cos \gamma;$$

$$\therefore c + b \cos \gamma = a \cot \theta \sin \gamma \cos \phi.$$

Also

$$b \sin \gamma = a \cot \theta \sin \gamma \sin \phi.$$

By squaring and adding, we have

$$c^2 + 2bc \cos \gamma + b^2 = a^2 \cot^2 \theta \sin^2 \gamma.$$

MISCELLANEOUS EXAMPLES. K.

1. If x is the number of degrees in the vertical angle $x + 12x + 12x = 180$, whence $x = \frac{180}{25} = 7.2$. Thus the angle is $7^\circ 12'$.

Again, the number of grades $= \frac{200}{180} \times \frac{180}{25} = 8$.

2. We have $\frac{\alpha}{4} = \frac{\beta}{5} = \frac{\gamma}{6} = \frac{\alpha + \beta + \gamma}{15} = \frac{\pi}{15}$,

whence $\alpha = \frac{4\pi}{15}$, $\beta = \frac{\pi}{3}$, $\gamma = \frac{2\pi}{5}$.

3. The first side $= \left(\frac{\cos A}{\sin A} - \frac{\sin A}{\cos A} \right) \div \left(\frac{\cos A}{\sin A} + \frac{\sin A}{\cos A} \right)$
 $= (\cos^2 A - \sin^2 A) \div (\cos^2 A + \sin^2 A)$
 $= \cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A$
 $= 1 - 2 \sin^2 A.$

$$4. \quad \cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{25}{169}} = \frac{12}{13};$$

$$\therefore \tan A + \sec A = \frac{5}{12} + \frac{13}{12} = \frac{3}{2}.$$

5. Let $AB=15$, $AC=30$, then since $\cos 60^\circ = \frac{1}{2}$, it is easy to see that CB is at right angles to BA .

$$\therefore CB = CA \sin 60^\circ = 30 \times \frac{\sqrt{3}}{2} = 25.98.$$

6. Let AB be the tower, BD the cliff, and C the point of observation; then if $BD=x$ ft., $CD=y$ ft., we have $50+x=y \tan \alpha$, $x=y \tan \beta$.

By division
$$1 + \frac{50}{x} = \frac{\tan \alpha}{\tan \beta} = \frac{1260}{1185} = \frac{84}{79}.$$

$$\therefore \frac{50}{x} = \frac{5}{79}, \text{ and } x=790.$$

7. If x ft. be the length of the arc, $\frac{x}{30}$ = radian measure of 10° ; whence

$$x = 30 \times \frac{\pi}{180} \times 10 = 5.236.$$

8. Here $\tan \alpha = \frac{8}{15}$; $\therefore \sin \alpha = \frac{8}{17}$, $\cos \alpha = \frac{15}{17}$.

9. Here $4 \sin^2 \theta - (2 + 2\sqrt{3}) \sin \theta + \sqrt{3} = 0$;
whence $(2 \sin \theta - \sqrt{3})(2 \sin \theta - 1) = 0.$

10. The expression
$$= \frac{5 \tan \alpha + 7}{6 - 3 \tan \alpha} = \frac{5 \times 4 + 7 \times 15}{6 \times 15 - 3 \times 4}$$
$$= \frac{5}{3} \times \frac{25}{26} = \frac{125}{78}.$$

11. First side
$$= 1 + 2(\sin A + \cos A) + (\sin A + \cos A)^2$$
$$= 1 + 2(\sin A + \cos A) + 1 + 2 \sin A \cos A$$
$$= 2 + 2(\sin A + \cos A) + 2 \sin A \cos A$$
$$= 2(1 + \sin A)(1 + \cos A).$$

12. The expression
$$= \sec^2 A (2 - \sec^2 A) - \operatorname{cosec}^2 A (2 - \operatorname{cosec}^2 A)$$
$$= (1 + \tan^2 A)(1 - \tan^2 A) - (1 + \cot^2 A)(1 - \cot^2 A)$$
$$= (1 - \tan^4 A) - (1 - \cot^4 A)$$
$$= (1 - \tan^4 A) - \left(1 - \frac{1}{\tan^4 A}\right)$$
$$= (1 - \tan^4 A) \left(1 + \frac{1}{\tan^4 A}\right) = \frac{1 - \tan^8 A}{\tan^4 A}.$$

$$\begin{aligned}
 13. \quad \sin \theta &= \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} = \frac{\sin \alpha - \cos \alpha}{\sqrt{(\sin \alpha - \cos \alpha)^2 + (\sin \alpha + \cos \alpha)^2}} \\
 &= \frac{\sin \alpha - \cos \alpha}{\sqrt{2} (\sin^2 \alpha + \cos^2 \alpha)} = \frac{\sin \alpha - \cos \alpha}{\sqrt{2}}.
 \end{aligned}$$

$$14. \quad \text{First side} = \left(\frac{1}{\sqrt{2}} - \frac{1}{2} \right) \div \left(\frac{1}{\sqrt{2}} + \frac{1}{2} \right) = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} = \frac{\sec 45^\circ - \tan 45^\circ}{\operatorname{cosec} 45^\circ + \cot 45^\circ}.$$

15. Since 9 degrees = 10 grades

$$1' = \frac{1}{60} \times \frac{10}{9} \times 100 \times 100 \times 1'' = \frac{10^4}{54} \times 1''.$$

$$\begin{aligned}
 16. \quad (2) \quad \text{Second side} &= \left(1 - \frac{\sin \theta}{\cos \theta} \right)^2 \div \left(1 - \frac{\cos \theta}{\sin \theta} \right)^2 \\
 &= \left(\frac{\cos \theta - \sin \theta}{\cos \theta} \right)^2 \div \left(\frac{\sin \theta - \cos \theta}{\sin \theta} \right)^2 \\
 &= \frac{\sec^2 \theta}{\operatorname{cosec}^2 \theta} = \frac{1 + \tan^2 \theta}{1 + \cot^2 \theta}.
 \end{aligned}$$

$$17. \quad (1) \quad \sin \theta + \frac{1}{\sin \theta} = \frac{3}{\sqrt{2}};$$

$$\therefore \sin^2 \theta - \frac{3}{\sqrt{2}} \sin \theta + 1 = 0,$$

$$\sqrt{2} \sin^2 \theta - 3 \sin \theta + \sqrt{2} = 0,$$

$$(\sqrt{2} \sin \theta - 1)(\sin \theta - \sqrt{2}) = 0;$$

whence $\sin \theta = \frac{1}{\sqrt{2}}$, and $\theta = 45^\circ$, the other value being impossible.

$$(2) \quad \cos \theta + \frac{1}{\cos \theta} = \frac{5}{2};$$

$$\therefore 2 \cos^2 \theta - 5 \cos \theta + 2 = 0,$$

$$(2 \cos \theta - 1)(\cos \theta - 2) = 0;$$

whence $\cos \theta = \frac{1}{2}$, and $\theta = 60^\circ$, the other value being impossible.

$$18. \quad \text{Radian measure of } 56^\circ = \frac{\pi}{180} \times 56 = \frac{22}{7} \times \frac{56}{180} = \frac{44}{45}.$$

$$\text{The arc traversed in } 36'' = \frac{36}{60} \times \frac{10}{60} \times 1760 \text{ yards} = 176 \text{ yards.}$$

$\therefore$ if d be the number of yards in the diameter,

$$\frac{176}{d} = \frac{22}{45}; \quad \therefore d = 360.$$

19. See Art. 35.

$$20. \quad (1) \quad \text{First side} = 1 \times \left(\frac{\sin^2 A}{\cos^2 A} - 1 \right) = \frac{\sin^2 A - \cos^2 A}{\cos^2 A} \\ = \sec^2 A (\sin^2 A - \cos^2 A) = \text{second side.}$$

$$(2) \quad \text{Second side} = \frac{\sin^2 \alpha \sin^2 \beta}{\sin^2 \beta} - \frac{\sin^2 \alpha \sin^2 \beta}{\sin^2 \alpha} \\ = \sin^2 \alpha - \sin^2 \beta \\ = \text{First side.} \quad [\text{Examples III. b, 34.}]$$

$$21. \quad \text{We have } \cos B = \frac{a}{c} = .405; \text{ and } a + c = 281.$$

$$\therefore c(1 + .405) = 281; \text{ whence } c = 200, a = 81.$$

$$\text{Also } b = \sqrt{c^2 - a^2} = \sqrt{33439} = 183 \text{ nearly.}$$

$$22. \quad \text{The radian measure} = \frac{\text{arc}}{\text{radius}} = \frac{495}{3 \times 1760} = \frac{3}{32} = .09375.$$

$$\text{With this as unit a right angle would be } \frac{\pi}{2} \div \frac{3}{32} = 16.7552.$$

$$23. \quad (1) \quad \text{First side} = \sin \theta \cos \theta \left\{ \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} \right\} = \cos^2 \theta + \sin^2 \theta = 1.$$

$$(2) \quad \text{First side} = \frac{\cot \theta}{\sec \theta} \times \frac{\cot^2 \theta}{\operatorname{cosec} \theta} \times \frac{\cos \theta}{\sin^3 \theta} \\ = \cot^3 \theta \times \frac{\cos \theta}{\sin \theta} \times \frac{\cos \theta}{\sin \theta} = \cot^5 \theta.$$

24. If O be the point of observation, $\angle BOA = 45^\circ$, and the line drawn from B perpendicular to OA bisects it at a point A' . Then

$$OA : OB = 2OA' : OB = 2 \cos 45^\circ = \sqrt{2} : 1.$$

$$25. \quad (1) \quad \text{First side} = (\sin^2 A + \operatorname{cosec}^2 A - 2) + (\cos^2 A + \sec^2 A - 2) \\ = (\sin^2 A + \cos^2 A) + (\operatorname{cosec}^2 A - 1) + (\sec^2 A - 1) - 2 \\ = \cot^2 A + \tan^2 A - 1.$$

$$(2) \quad \text{First side} = 3 \cot^2 \theta - 10 \cot \theta + 3 = 3 \operatorname{cosec}^2 \theta - 10 \cot \theta.$$

$$27. \quad \text{The expression} = \frac{2 - \cot A}{2 + 3 \cot A} = \frac{2 - \frac{9}{2}}{2 + \frac{27}{2}} = -\frac{5}{31}.$$

$$28. \quad 2 \cos \theta \cot \theta + 1 - \cot \theta - 2 \cos \theta = 0;$$

$$\therefore 2 \cos \theta (\cot \theta - 1) - (\cot \theta - 1) = 0;$$

$$\therefore (2 \cos \theta - 1)(\cot \theta - 1) = 0;$$

$$\therefore \theta = 60^\circ, \text{ or } 45^\circ.$$

$$29. \quad \text{If } x \text{ inches be the length of the arc, } \frac{x}{6} = \frac{\pi}{180} \times \frac{1217}{60}.$$

$$\text{In the second circle } \theta = \frac{\pi}{180} \times \frac{1217}{10} \times \frac{1}{8};$$

$$\therefore \text{sexagesimal measure of } \theta = \frac{1217}{10 \times 8} = 15^\circ 12' 45''.$$

30. Take the figure of the Example on p. 41, and let

$$PT = x \text{ yards, } RT = y \text{ yards.}$$

Then $x = y + 110$, since $\angle PQT = 45^\circ$.

Also

$$y = x \cot 60^\circ = \frac{x}{\sqrt{3}}.$$

$$\therefore x \left(\frac{\sqrt{3} - 1}{\sqrt{3}} \right) = 110;$$

$$\therefore x = 55(3 + \sqrt{3}) = 260.26.$$

$$\begin{aligned} 31. \quad \text{First side} &= \frac{1 + \cos A}{1 - \cos A} - \frac{1 - \cos A}{1 + \cos A} - 4 \cot^2 A \\ &= \frac{(1 + \cos A)^2 - (1 - \cos A)^2}{(1 - \cos A)(1 + \cos A)} - 4 \cot^2 A \\ &= \frac{2 \cdot 2 \cos A}{\sin^2 A} - \frac{4 \cos^2 A}{\sin^2 A} = \frac{4 \cos A (1 - \cos A)}{1 - \cos^2 A} \\ &= \frac{4 \cos A}{1 + \cos A} = \frac{4}{1 + \sec A}. \end{aligned}$$

$$32. \quad (1) \quad 8(1 - \cos^2 \theta) - 2 \cos \theta = 5;$$

$$8 \cos^2 \theta + 2 \cos \theta - 3 = 0;$$

$$(2 \cos \theta - 1)(4 \cos \theta + 3) = 0;$$

$$\text{whence } \theta = 60^\circ; \text{ or } \cos \theta = -\frac{3}{4}.$$

$$(2) \quad 5 \tan^2 x - (1 + \tan^2 x) = 11;$$

$$4 \tan^2 x = 12;$$

$$\tan x = \pm \sqrt{3}.$$

From the first of these values $x = 60^\circ$.

33. Here
$$\theta = \frac{\text{arc}}{\text{radius}} = \frac{11}{5 \times 12};$$

$\therefore$ the angle in degrees $= \frac{180}{\pi} \times \frac{11}{60} = \frac{180}{22} \times \frac{11 \times 7}{60} = 10\frac{1}{2}.$

35. See Art. 16.

(2) Here
$$\cos \theta = \frac{2ab}{a^2 + b^2}.$$

Now since $(a - b)^2$ is a positive quantity, $a^2 + b^2 > 2ab$; $\therefore \cos \theta < 1$, which is possible.

36. Let ACB be the hill, A being the summit and C a point halfway down. Draw AD , CE perpendicular to the horizontal line through the object O . Then $AD = 2CE$, and $BD = 2BE$.

Now $OB + BE = CE \cot \beta$, and $OB + 2BE = 2CE \cot \alpha$;

therefore, by subtraction, $\frac{OB + 2BE}{CE} - \frac{OB + BE}{CE} = 2 \cot \alpha - \cot \beta$;

that is, $\frac{BE}{CE} = 2 \cot \alpha - \cot \beta$; also $\frac{BE}{CE} = \cot \theta$.

37. From the figure in Ex. 2, Art. 46, we have

$$\cos B = \frac{a}{c} = \frac{25\sqrt{2}}{50} = \frac{1}{\sqrt{2}}; \therefore B = 45^\circ;$$

$$\therefore b = c \sin 45^\circ = 50 \times \frac{1}{\sqrt{2}} = 25\sqrt{2}.$$

If p be the perpendicular from C , $p = a \sin 45^\circ = 25$.

40. Since $\tan \theta = \pm 1$, the angles will be those coterminal with 45° , 135° , 225° , 315° .

41. Take the figure of Example II. on p. 43.

Let $AE = x$, $CE = y$; then

$$\frac{x}{y} = .965; \quad \frac{x + 42}{y} = 1.6.$$

$$\therefore \frac{x + 42}{x} = \frac{1600}{965} = 1 + \frac{635}{965};$$

whence $x = 42 \times \frac{965}{635} = 63$, approximately;

$$\therefore AB = 63 + 42 = 105.$$

42. We have $\tan 15^\circ = \frac{\sin 30^\circ}{1 + \cos 30^\circ} = \frac{1}{2} \bigg/ \left(1 + \frac{\sqrt{3}}{2}\right) = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}.$

Similarly

$$\tan 75^\circ = 2 + \sqrt{3}.$$

Again,

$$1 + \tan^2 \theta = 4 \tan \theta,$$

$$\tan^2 \theta - 4 \tan \theta + 1 = 0;$$

$$\therefore \tan \theta = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3};$$

$$\therefore \theta = 75^\circ, \text{ or } 15^\circ.$$

43. The first side

$$\begin{aligned} &= 1 + \sec \theta + \tan \theta + \operatorname{cosec} \theta + \sec \theta \operatorname{cosec} \theta \\ &\quad + \operatorname{cosec} \theta \tan \theta + \cot \theta + \sec \theta \cot \theta + 1 \\ &= 2 + \sec \theta + \tan \theta + \operatorname{cosec} \theta + \sec \theta \operatorname{cosec} \theta + \sec \theta + \cot \theta + \operatorname{cosec} \theta \\ &= 2(1 + \sec \theta + \operatorname{cosec} \theta) + \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} + \frac{1}{\sin \theta \cos \theta} \\ &= 2(1 + \sec \theta + \operatorname{cosec} \theta) + \frac{2(\sin^2 \theta + \cos^2 \theta)}{\sin \theta \cos \theta} \\ &= 2(1 + \sec \theta + \operatorname{cosec} \theta + \tan \theta + \cot \theta). \end{aligned}$$

44. See figure and notation of Art. 25.

45. We have $2(1 - \sin^2 \theta) = 1 + \sin \theta;$
 $\therefore (1 + \sin \theta)(1 - 2 \sin \theta) = 0;$
 $\therefore \sin \theta = -1, \text{ or } \sin \theta = \frac{1}{2};$

whence

$$\theta = 30^\circ, 150^\circ, 270^\circ.$$

46 $\sin(270^\circ + A) = -\sin(90^\circ + A) = -\cos A.$

But

$$\begin{aligned} \cos A &= \pm \sqrt{1 - \sin^2 A} = \pm .8; \\ \therefore \sin(270^\circ + A) &= \pm .8. \end{aligned}$$

47. See Art. 113.

48. The expression $= \frac{2 \sin 2A \sin A}{2 \sin A \cos 2A} = \tan 2A.$ See Art. 89.

49. $\tan A = \sqrt{\sec^2 A - 1} = \pm \sqrt{\frac{4}{3} - 1} = \pm \frac{1}{\sqrt{3}}.$

The angle is coterminal with 150° or 210° , and the tangents of these angles are equal but opposite in sign.

50. Take the figure of Example on p. 41, and let

$$PT = x \text{ yards, } \angle PQT = 45^\circ, \angle PRT = 60^\circ.$$

Also

$$QR = 1760, \text{ and } QT = PT;$$

$$\therefore \frac{x}{x - 1760} = \tan 69^\circ = \sqrt{3};$$

$$\therefore x(\sqrt{3} - 1) = 1760\sqrt{3};$$

$$\therefore x = 880(3 + \sqrt{3}) = 4164.16.$$

$$\begin{aligned} 52. \quad (1) \quad \text{First side} &= \sin^2 a (\sin^2 a + 2 \cos^2 a) \\ &= (1 - \cos^2 a)(1 + \cos^2 a) = 1 - \cos^4 a. \end{aligned}$$

$$\begin{aligned} (2) \quad \text{First side} &= \sec 2 \left(\frac{\pi}{4} - \theta \right) \quad [\text{Art. 124}] \\ &= \operatorname{cosec} 2\theta. \end{aligned}$$

$$\begin{aligned} (3) \quad \cos 10^\circ + \sin 40^\circ &= \cos 10^\circ + \cos 50^\circ = 2 \cos 30^\circ \cos 20^\circ \\ &= \sqrt{3} \sin 70^\circ. \end{aligned}$$

$$53. \quad \text{The expression} = \frac{a \tan \theta - b}{a \tan \theta + b} = \frac{a^2 - b^2}{a^2 + b^2}.$$

54. Multiply all through by $\cos 18^\circ$, then we have to prove that

$$4 \cos^2 18^\circ - 3 = 2 \sin 18^\circ.$$

$$\begin{aligned} \text{First side} &= 4 \left\{ 1 - \left(\frac{\sqrt{5} - 1}{4} \right)^2 \right\} - 3 = \frac{10 + 2\sqrt{5}}{4} - 3 \\ &= \frac{\sqrt{5} - 1}{2} = 2 \sin 18^\circ. \end{aligned}$$

$$56. \quad \text{Here } \theta = \frac{\text{arc}}{\text{radius}} = \frac{20 \times 10}{60 \times 60} \div \frac{1}{2} = \frac{1}{9};$$

$$\therefore D = \frac{1}{9} \times 180 \times \frac{7}{22} = \frac{70}{11} = 6\frac{4}{11}^\circ.$$

$$57. \quad \text{The expression} = \frac{2 - 3 \cot a}{4 - 9 \tan a}, \text{ where } \tan a = \frac{12}{5},$$

$$\begin{aligned} &= \frac{2 - 3 \times \frac{5}{12}}{4 - 9 \times \frac{12}{5}} = \frac{3}{4} \div \left(-\frac{88}{5} \right) \\ &= -\frac{15}{352}. \end{aligned}$$

$$\begin{aligned}
 58. \quad (2) \quad \text{First side} &= \frac{2 \sin \frac{\pi}{4} \sin \theta}{2 \cos \frac{2\pi}{3} \sin \theta} + \sqrt{2} \\
 &= \frac{1}{\sqrt{2}} \left(-\frac{1}{2} \right) + \sqrt{2} = 0.
 \end{aligned}$$

$$\begin{aligned}
 59. \quad (1) \quad \text{The expression} &= \frac{\cos A \cos C + (-\cos C)(-\cos A)}{\cos A \sin C - \sin C(-\cos A)} \\
 &= \frac{2 \cos A \cos C}{2 \cos A \sin C} = \cot C.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \text{The expression} &= \frac{\sin A \cos A + \text{two similar terms}}{\sin A \sin B \sin C} \\
 &= \frac{1}{2} \left(\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A \sin B \sin C} \right) \\
 &= 2. \quad [\text{Art. 135, Ex. 1.}]
 \end{aligned}$$

60. Let ABC be the horizontal equilateral triangle, and let PQ be the flagstaff. Then since each side subtends an angle of 60° at P , the top of the flagstaff, the triangles PCB , PBA , PAC are equilateral.

Let x be a side of $\triangle ABC$;

then
$$AQ = \frac{x}{2 \sec 30^\circ} = \frac{x}{\sqrt{3}}.$$

Then from $\triangle PAQ$,

$$PA^2 = AQ^2 + QP^2,$$

$$x^2 = \frac{x^2}{3} + 10000;$$

whence

$$x^2 = 15000, \text{ or } x = 50\sqrt{6}.$$

$$\begin{aligned}
 61. \quad \text{The first expression} &= \sin \theta + \cos \theta + \sin^2 \theta + \cos^2 \theta \\
 &= \sin \theta + \cos \theta + 1.
 \end{aligned}$$

Similarly, the second expression $= \sin \theta + \cos \theta - 1$;

$$\therefore \text{ the product} = (\sin \theta + \cos \theta)^2 - 1 = 2 \sin \theta \cos \theta = \sin 2\theta.$$

$$\begin{aligned}
 62. \quad \text{Second side} &= \sin^2 \frac{\theta + \phi}{2} + \cos^2 \frac{\theta - \phi}{2} - 2 \sin \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} \\
 &= \frac{1}{2} \{1 - \cos(\theta + \phi) + 1 + \cos(\theta - \phi)\} - \sin \theta - \sin \phi. \\
 &= 1 - \sin \theta - \sin \phi + \frac{1}{2} \{\cos(\theta - \phi) - \cos(\theta + \phi)\} \\
 &= 1 - \sin \theta - \sin \phi + \sin \theta \sin \phi \\
 &= (1 - \sin \theta)(1 - \sin \phi).
 \end{aligned}$$

63. The expression $= \frac{2 \cos \alpha \sin \theta}{2 \sin \beta \sin \theta} = \frac{\cos \alpha}{\sin \beta}$, which is independent of θ .

64. First side $= \sin \frac{B+C}{2} \cos \frac{B-C}{2} + \dots + \dots$

$$= \frac{1}{2} \{ (\sin B + \sin C) + \dots + \dots \}$$

$$= \sin A + \sin B + \sin C.$$

65. We have $\frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \frac{1 - \sin^2 \theta}{\sin \theta} = \frac{\cos^2 \theta}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}};$

$$2 \sin^2 \frac{\theta}{2} = \left(2 \cos^2 \frac{\theta}{2} - 1 \right)^2;$$

$$\therefore 4 \cos^4 \frac{\theta}{2} - 2 \cos \frac{\theta}{2} - 1 = 0;$$

$$\therefore \cos^2 \frac{\theta}{2} = \frac{2 \pm \sqrt{20}}{8} = \frac{1 \pm \sqrt{5}}{4};$$

$\therefore \cos^2 \frac{\theta}{2} = \cos 36^\circ$, the other value being impossible.

66. Draw ZW perpendicular to XY and let $ZW = x$. Then $\triangle XZY$ is right-angled at Z .

And $XZ = XY \cos 30^\circ = 100\sqrt{3}$ yds.

Again, from $\triangle WXZ$, $WZ = XZ \sin 30^\circ = 50\sqrt{3} = 86.6$ yds.

67. We have $32\pi \times 1000 = \frac{5585}{2} \times 3 \times 12$; whence $\pi = 3.141$, approximately.

68. $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = \frac{1 - \cos 2\alpha}{2} + \dots + \dots$

$$= \frac{3}{2} - \frac{1}{2} (\cos 2\alpha + \cos 2\beta + \cos 2\gamma)$$

$$= \frac{3}{2} - \frac{1}{2} \{ 2 \cos (\alpha + \beta) \cos (\alpha - \beta) + 1 - 2 \sin^2 \gamma \}$$

$$= 1 - \{ \sin \gamma \cos (\alpha - \beta) - \sin^2 \gamma \}$$

$$= 1 - \sin \gamma \{ \cos (\alpha - \beta) - \cos (\alpha + \beta) \}$$

$$= 1 - 2 \sin \alpha \sin \beta \sin \gamma.$$

$$69. \quad (1) \quad \text{First side} = \left(\frac{\sin A}{\cos A} + \frac{\sin 2A}{\cos 2A} \right) (2 \cos 2A \cos A) \\ = 2 (\sin 2A \cos A + \cos 2A \sin A) = 2 \sin 3A.$$

(2) Multiply all through by 32 ; then

$$\begin{aligned} \text{Second side} &= 2 + \cos 2A - 2 \cos 4A - \cos 6A \\ &= 2 (1 - \cos 4A) + 2 \sin 4A \sin 2A \\ &= 4 \sin^2 2A + 4 \sin^2 2A \cos 2A \\ &= 4 \sin^2 2A (1 + \cos 2A) \\ &= 16 \sin^2 A \cos^2 A \cdot 2 \cos^2 A \\ &= 32 \sin^2 A \cos^4 A. \end{aligned}$$

$$70. \quad \text{The expression} = \frac{2 \cos 13a \sin 10a}{2 \sin 10a \cos 6a} = \frac{\cos 13a}{\cos 6a} \\ = \frac{\cos 13a}{\cos (\pi - 13a)} = -1.$$

71. We have $\cot (A+B) = 1$. Therefore

$$\begin{aligned} \cot A \cot B - 1 &= \cot A + \cot B; \\ \therefore 2 \cot A \cot B &= 1 + \cot A + \cot B + \cot A \cot B \\ &= (1 + \cot A) (1 + \cot B); \\ \therefore \frac{\cot A}{1 + \cot A} \cdot \frac{\cot B}{1 + \cot B} &= \frac{1}{2}. \end{aligned}$$

72. For the first part see XI. d. Ex. 15.

Then

$$\begin{aligned} \tan \theta &= \cot \theta - 2 \cot 2\theta, \\ 2 \tan 2\theta &= 2 \cot 2\theta - 4 \cot 4\theta, \\ 4 \tan 4\theta &= 4 \cot 4\theta - 8 \cot 8\theta; \end{aligned}$$

$\therefore$ by addition

$$\tan \theta + 2 \tan 2\theta + 4 \tan 4\theta = \cot \theta - 8 \cot 8\theta.$$

$$73. \quad \text{The expression} = 1 - \frac{\sin^3 \theta}{\sin \theta + \cos \theta} - \frac{\cos^3 \theta}{\sin \theta + \cos \theta} \\ = 1 - \frac{\sin^3 \theta + \cos^3 \theta}{\sin \theta + \cos \theta} \\ = 1 - (\sin^2 \theta + \cos^2 \theta - \sin \theta \cos \theta) \\ = \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta.$$

$$74. \quad \text{We have } x = 3 \sin A - (3 \sin A - 4 \sin^3 A) = 4 \sin^3 A, \\ y = 4 \cos^3 A - 3 \cos A + 3 \cos A = 4 \cos^3 A;$$

$$\therefore \left(\frac{x}{4}\right)^{\frac{2}{3}} + \left(\frac{y}{4}\right)^{\frac{2}{3}} = \sin^2 A + \cos^2 A = 1; \\ \therefore x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4^{\frac{2}{3}}.$$

75. Let PQ, RS be the flagstaffs of lengths x, y feet respectively. Then $QABS$ is a straight line, such that $AB = 30$ ft., $\angle PAQ = 60^\circ$, $\angle RAS = 30^\circ$, $\angle PBQ = 45^\circ$, $\angle RBS = 60^\circ$. Let $AQ = a$, $BS = b$. Then since

$$\angle PBQ = 45^\circ, \quad BQ = QP = x; \\ \therefore a = x - 30.$$

From $\triangle ARS$, we have $AS = RS \cot 30^\circ = y\sqrt{3}$;

$$\therefore b = y\sqrt{3} - 30.$$

From $\triangle APQ$, $PQ = AQ \tan 60^\circ$;

$$\therefore x = (x - 30)\sqrt{3};$$

whence

$$x = 15(3 + \sqrt{3}).$$

From $\triangle BRS$, $RS = BS \tan 60^\circ$;

$$\therefore y = \sqrt{3}(y\sqrt{3} - 30);$$

whence

$$y = 15\sqrt{3}.$$

Again,

$$QS = a + b + 30 \\ = x + y\sqrt{3} - 30 \\ = 45 + 15\sqrt{3} + 45 - 30 \\ = 60 + 15\sqrt{3}.$$

76. (1) First side

$$= \frac{1}{2}(1 + \cos 2A) + \frac{1}{2}(1 + \cos 2B) - 2 \cos A \cos B \cos(A + B) \\ = 1 + \cos(A + B) \cos(A - B) - 2 \cos A \cos B \cos(A + B) \\ = 1 + \cos(A + B) \{\sin A \sin B - \cos A \cos B\} \\ = 1 - \cos^2(A + B) = \sin^2(A + B).$$

(2) First side

$$= 2(\sin 5A - \sin A) - (\sin 3A + \sin A) \\ = 4 \cos 3A \sin 2A - 2 \sin 2A \cos A \\ = 2 \sin 2A (2 \cos 3A - \cos A) \\ = 4 \sin A \cos A (8 \cos^3 A - 7 \cos A) \\ = 4 \sin A \cos^2 A \{8(1 - \sin^2 A) - 7\} \\ = 4 \sin A \cos^2 A (1 - 8 \sin^2 A).$$

77. If r is the radius of the circle, and AB one side of the square, we have $2\pi r = 3$, and

$$\begin{aligned} AB &= 2r \sin 45^\circ = \frac{3\sqrt{2}}{2\pi} \\ &= \frac{3}{2} \times 1.4142 \times .3183 \\ &= .6752 \text{ feet} \\ &= 8.10 \text{ inches.} \end{aligned}$$

78. Here $AB = 2r \sin 54^\circ$, $BC = 2r \sin 30^\circ$, $CD = 2r \sin 18^\circ$. And it remains to prove that

$$\sin 54^\circ = \sin 30^\circ + \sin 18^\circ.$$

[See Examples XI. *c.* 9.]

79. First side $= (2 + \sqrt{3}) + (2 - \sqrt{3}) - 1 - 2 = 1$.

80. We have, by addition, $\cot \theta = 2(m + n)$.

Also, by subtraction, $\cos \theta = 2(m - n)$.

$$\therefore 4(m^2 - n^2) = \frac{\cos^2 \theta}{\sin \theta};$$

$$\begin{aligned} \therefore 16(m^2 - n^2)^2 &= \frac{\cos^4 \theta}{\sin^2 \theta} = \cot^2 \theta \times \cos^2 \theta \\ &= \cot^2 \theta (1 - \sin^2 \theta) \\ &= 16mn. \end{aligned}$$

81. (1) First side $= \frac{1}{2} [\sin (2\beta + \alpha) + \sin \alpha - \sin (2\gamma + \alpha) - \sin \alpha]$
 $= \frac{1}{2} [2 \cos (\alpha + \beta + \gamma) \sin (\beta - \gamma)].$

(2) First side $= \left(\frac{\sin 2A}{\cos 2A} - \frac{\sin A}{\cos A} \right) \left(\frac{1}{\cos A} + \frac{1}{\cos 3A} \right)$
 $= \frac{\sin A}{\cos 2A \cos A} \cdot \frac{\cos 3A + \cos A}{\cos A \cos 3A}$
 $= \frac{2 \sin A \cos 2A \cos A}{\cos 2A \cos^2 A \cos 3A} = 2 \sin A \sec A \sec 3A.$

82. $2 \cos 6^\circ \cos 66^\circ = \cos 72^\circ + \cos 60^\circ = \sin 18^\circ + \frac{1}{2}$
 $= \frac{\sqrt{5} - 1}{4} + \frac{1}{2} = \frac{\sqrt{5} + 1}{4}.$

$$\begin{aligned}
 2 \cos 42^\circ \cos 78^\circ &= \cos 120^\circ + \cos 36^\circ \\
 &= -\frac{1}{2} + \frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}-1}{4}.
 \end{aligned}$$

$$\therefore 4 \cos 6^\circ \cos 66^\circ \cos 42^\circ \cos 78^\circ = \frac{1}{4}.$$

83. Put $2A = 45^\circ$.

84. The distance required is evidently equal to $10 \tan 22\frac{1}{2}^\circ$

$$= \frac{10}{\sqrt{2}+1} = 10(\sqrt{2}-1) = 4.14 \text{ miles.}$$

85. (1) Separate each term into the difference of two cosines.

(2) Second side = $\frac{\sin \theta + 2 \sin \theta \cos \theta}{\cos \theta + 2 \cos^2 \theta}$

$$= \frac{\sin \theta (1 + 2 \cos \theta)}{\cos \theta (1 + 2 \cos \theta)} = \tan \theta.$$

86. First side = $2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} + 2 \cos^2 \frac{\gamma}{2} - 1$

$$\begin{aligned}
 &= 2 \cos \frac{\gamma}{2} \cos \frac{\alpha-\beta}{2} + 2 \cos^2 \frac{\gamma}{2} - 1 \\
 &= 2 \cos \frac{\gamma}{2} \left\{ \cos \frac{\alpha-\beta}{2} + \cos \frac{\gamma}{2} \right\} - 1 \\
 &= 2 \cos \frac{\gamma}{2} \left\{ \cos \frac{\alpha-\beta}{2} + \cos \frac{\alpha+\beta}{2} \right\} - 1 \\
 &= 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} - 1.
 \end{aligned}$$

87. Put $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$, then

$$\begin{aligned}
 \text{first side} &= \frac{k^2 (\sin^2 B - \sin^2 C)}{k \sin A} \cos A + \dots + \dots \\
 &= k \left\{ \frac{\sin (B+C) \sin (B-C)}{\sin A} \cos A + \dots + \dots \right\} \\
 &= k \{ \sin (B-C) \cos A + \dots + \dots \} \\
 &= -k \{ \sin (B-C) \cos (B+C) + \dots + \dots \} \\
 &= -\frac{k}{2} \{ (\sin 2B - \sin 2C) + \dots + \dots \} \\
 &= 0.
 \end{aligned}$$

$$\begin{aligned}
 88. \quad \text{First side} &= a \cos 2\theta + b \sin 2\theta \\
 &= a(1 - 2 \sin^2 \theta) + 2b \sin \theta \cos \theta \\
 &= a + 2 \sin \theta (b \cos \theta - a \sin \theta) \\
 &= a, \text{ since } b \cos \theta = a \sin \theta.
 \end{aligned}$$

$$89. \quad \text{Let } \log_a b = x, \text{ so that } a^x = b,$$

$$\log_b c = y, \dots\dots\dots b^y = c,$$

$$\log_c a = z, \dots\dots\dots c^z = a.$$

$$\text{Then we have} \qquad a = c^z = b^{yz} = a^{xyz};$$

$$\therefore xyz = 1, \text{ or } \log_a b \log_b c \log_c a = 1.$$

$$\text{We have } \log 8 = \log 2^3 = 3 \log 2; \text{ whence } \log 2 = \cdot 30103;$$

$$\begin{aligned}
 \log 2 \cdot 4 &= \log \left(\frac{3 \times 8}{10} \right) = \log 3 + \log 8 - 1 \\
 &= \bar{1} \cdot 47712 + \cdot 90309 = \cdot 38021.
 \end{aligned}$$

$$\begin{aligned}
 \log 5400 &= 2 + \log 2 + 3 \log 3 \\
 &= 2 \cdot 30103 + 1 \cdot 43136 = 3 \cdot 73239.
 \end{aligned}$$

$$\begin{aligned}
 L \tan 30^\circ &= 10 + \log \frac{1}{\sqrt{3}} = 10 - \frac{1}{2} \log 3 \\
 &= 9 \cdot 76144.
 \end{aligned}$$

$$90. \quad \cot(A+B) = \cot(90^\circ - C) = \tan C = \frac{1}{\cot C};$$

$$\therefore \frac{\cot A \cot B - 1}{\cot A + \cot B} = \frac{1}{\cot C};$$

whence by multiplying up and rearranging we obtain the required result.

For the second part, put $A = 15^\circ$, $B = 30^\circ$, $C = 45^\circ$.

$$\begin{aligned}
 91. \quad \text{First side} &= (1 + \sin 2A)^2 + \cos^2 2A + 2 \cos 2A (1 + \sin 2A) \\
 &= (1 + \sin 2A)^2 + (1 - \sin^2 2A) + 2 \cos 2A (1 + \sin 2A) \\
 &= (1 + \sin 2A) \{1 + \sin 2A + 1 - \sin 2A + 2 \cos 2A\} \\
 &= 2(1 + \cos 2A)(1 + \sin 2A) = 4 \cos^2 A (1 + \sin 2A).
 \end{aligned}$$

$$92. \quad \text{See Art. 150.}$$

$$93. \quad \text{We have to prove that } \sin 9^\circ \sin 81^\circ = \sin 12^\circ \sin 48^\circ.$$

$$\begin{aligned}
 \text{First side} &= \frac{1}{2} (\cos 72^\circ - \cos 90^\circ) \\
 &= \frac{1}{2} \sin 18^\circ = \frac{\sqrt{5} - 1}{8}.
 \end{aligned}$$

$$\begin{aligned}\text{Second side} &= \frac{1}{2} (\cos 36^\circ - \cos 60^\circ) \\ &= \frac{1}{2} \left(\frac{\sqrt{5}+1}{4} - \frac{1}{2} \right) = \frac{\sqrt{5}-1}{8}.\end{aligned}$$

94. See Art. 136, Ex. 2.

$$95. \quad L \sin \theta > L \sin 27^\circ 45' \text{ by } \frac{1742}{2400} \times 60'';$$

$$\text{whence} \quad \theta = 27^\circ 45' 44''.$$

96. By a well-known algebraical formula,

$$x^3 + y^3 + z^3 = 3xyz,$$

$$\text{when} \quad x + y + z = 0;$$

therefore we have

$$\cos^3 A + \cos^3 B + \cos^3 C = 3 \cos A \cos B \cos C.$$

Substituting $\frac{1}{4}(\cos 3A + 3 \cos A)$ for $\cos^3 A$, and similar results for $\cos^3 B$, $\cos^3 C$,

we have

$$\frac{1}{4}(\cos 3A + \cos 3B + \cos 3C) + \frac{3}{4}(\cos A + \cos B + \cos C) = 3 \cos A \cos B \cos C,$$

whence the required result follows at once, since

$$\cos A + \cos B + \cos C = 0.$$

97. $\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{49}{625}} = \pm \frac{24}{25}$; but since A lies between 270° and 360° , we must reject the negative value; thus $\cos A = \frac{24}{25}$.

$$\begin{aligned}\text{Hence } \sin 2A &= 2 \sin A \cos A = 2 \left(-\frac{7}{25} \right) \left(\frac{24}{25} \right) \\ &= -\frac{336}{625}.\end{aligned}$$

$$\begin{aligned}\text{Also } \tan \frac{A}{2} &= \frac{1 - \cos A}{\sin A} = \operatorname{cosec} A - \cot A \\ &= -\frac{25}{7} + \frac{24}{7} = -\frac{1}{7}.\end{aligned}$$

98. We have
$$\frac{2 \cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = \left(\frac{\sin \theta + \cos \theta}{\sin \theta} \right)^2 ;$$

$$8 \cos^3 \frac{\theta}{2} \sin \frac{\theta}{2} = 1 + \sin 2\theta,$$

$$4 \cos^2 \frac{\theta}{2} \sin \theta = 1 + \sin 2\theta,$$

$$2 \sin \theta (1 + \cos \theta) = 1 + \sin 2\theta ;$$

$$\therefore 2 \sin \theta = 1,$$

or
$$\theta = 30^\circ.$$

Again, by putting $\theta = 30^\circ$, we have

$$2 \cot 15^\circ = (1 + \sqrt{3})^2 = 4 + 2\sqrt{3}.$$

$$\therefore \cot 15^\circ = 2 + \sqrt{3}, \text{ and } \tan 15^\circ = 2 - \sqrt{3}.$$

99. We have $\log 360 = 2 \log 2 + 2 \log 3 + 1.$

$$\therefore 2 \log 3 = \log 360 - 2 \log 2 - 1$$

$$= 1.5563025 - .6020600$$

$$= .9542425 ;$$

$$\therefore \log 3 = .4771213.$$

Now $\log .04 = \log 4 - 2 = 2 \log 2 - 2 = \bar{2}.60206.$

$$\log 24 = 3 \log 2 + \log 3 = .90309 + .4771213$$

$$= 1.3802113.$$

$$\log 6 = \log \frac{2}{3} = \log 2 - \log 3$$

$$= .30103 - .4771213 = \bar{1}.8239087.$$

Again let $\log_2 30 = x$, so that $2^x = 30.$

$$\therefore x \log 2 = \log 30 = 1 + \log 3 ;$$

$$\therefore x = \frac{1 + \log 3}{\log 2} = \frac{1.4771213}{.30103} = 4.90689.$$

100. This follows at once from Art. 134, Ex. 5.

$$\begin{aligned}
 101. \quad \cot(a + \beta) &= \frac{\cot a \cot \beta - 1}{\cot a + \cot \beta} = \frac{x(x + x^{-1} + 1) - 1}{(x + x^{-1} + 1)^{\frac{1}{2}}(1 + x)} = \frac{x^2 + x}{(1 + x)(x + x^{-1} + 1)^{\frac{1}{2}}} \\
 &= \frac{x}{(x + x^{-1} + 1)^{\frac{1}{2}}} = \frac{1}{x^{-1}(x + x^{-1} + 1)^{\frac{1}{2}}} = \frac{1}{(x^{-1} + x^{-3} + x^{-2})^{\frac{1}{2}}} \\
 &= \cot \gamma.
 \end{aligned}$$

Therefore $a + \beta = \gamma$.

102. Let A and a be given, and let B be the right angle; then $c = a \cot A$, $b = a \operatorname{cosec} A$, or $b = \sqrt{a^2 + c^2}$. Also $C = 90^\circ - A$.

If $A = 31^\circ 53' 26.8''$, $a = 28$, we have

$$\begin{aligned}
 c &= 28 \cot A. \\
 \log c &= \log 28 + \log \cot 31^\circ 53' 26.8'', \\
 \log 28 &= 1.4471580 \\
 \log \cot 31^\circ 53' &= .2061805 \\
 \text{decrease for } 26.8'' & \quad \quad \quad 1258 \\
 \therefore \log c &= 1.6532127; \therefore c = 45.
 \end{aligned}$$

Again

$$\begin{aligned}
 b^2 &= a^2 + c^2 = 2025 + 784 = 2809; \\
 \therefore b &= 53; \text{ also } C = 90^\circ - A = 58^\circ 6' 33.2''.
 \end{aligned}$$

104. The greatest angle, C , is opposite to $\sqrt{x^2 + xy + y^2}$.

$$\begin{aligned}
 \therefore \cos C &= \frac{x^2 + y^2 - (x^2 + xy + y^2)}{2xy} = -\frac{1}{2}; \\
 \therefore C &= 120^\circ.
 \end{aligned}$$

$$\begin{aligned}
 105. \quad \cos 3A + \sin 3A &= 4 \cos^3 A - 3 \cos A + 3 \sin A - 4 \sin^3 A \\
 &= 4(\cos^3 A - \sin^3 A) - 3(\cos A - \sin A),
 \end{aligned}$$

which is evidently divisible by $\cos A - \sin A$.

See solution to Examples XII. c, 27.

$$\begin{aligned}
 106. \quad \text{We have} \quad \cot \frac{C}{2} &= \tan \frac{A+B}{2} = \frac{\frac{5}{6} + \frac{20}{37}}{1 - \frac{5}{6} \cdot \frac{20}{37}} = \frac{5}{2}. \\
 \therefore \tan C &= \frac{2 \tan \frac{C}{2}}{1 - \tan^2 \frac{C}{2}} = \frac{2 \times \frac{5}{2}}{1 - \frac{25}{25}} = \frac{20}{21}.
 \end{aligned}$$

For the second part, it will be sufficient to prove that

$$\sin A + \sin C = 2 \sin B.$$

Now $\sin A = \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} = \frac{60}{61}$, on reduction.

Similarly $\sin C = \frac{20}{29}$, and $\sin B = \frac{1480}{1769}$, whence the result follows.

107. The first factor easily reduces to $2 \cot 2\theta$, and the second to $\frac{2}{\cos 2\theta}$; whence the product becomes $4 \operatorname{cosec} 2\theta$.

108. $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{128}{603}}$,

$\log \tan \frac{A}{2} = \frac{1}{2} \{\log 128 - \log 603\} = \bar{1} \cdot 6634465$;

whence $\frac{A}{2} = 24^\circ 44' 16''$, and $A = 49^\circ 28' 32''$.

109. $\tan(A-B) = \frac{\tan A - \frac{n \sin A \cos A}{1 - n \sin^2 A}}{1 + \tan A \left(\frac{n \sin A \cos A}{1 - n \sin^2 A} \right)} = \frac{\tan A \left(1 - \frac{n \cos^2 A}{1 - n \sin^2 A} \right)}{1 + \frac{n \sin^2 A}{1 - n \sin^2 A}}$
 $= (1-n) \tan A$.

110. $\log 200 = 2 + \log 2 = 3 - \log 5 = 2 \cdot 30103$.

$\log \cdot 025 = 2 \log 5 - 3 = \bar{2} \cdot 39794$,

$\log \sqrt[3]{62 \cdot 5} = \frac{1}{3} (\log 625 - 1) = \frac{1}{3} (4 \log 5 - 1)$
 $= \cdot 598626$.

$L \sin 30^\circ = 10 + \log \left(\frac{1}{2} \right) = 10 + \log 5 - 1$
 $= 9 \cdot 69897$.

$L \cos 45^\circ = 10 + \log \left(\frac{1}{\sqrt{2}} \right) = 10 + \frac{1}{2} (\log 5 - 1)$
 $= 9 \cdot 849485$.

111. (1) $\frac{\cot(A-30^\circ)}{\tan(A+30^\circ)} = \frac{\cos(A-30^\circ) \cos(A+30^\circ)}{\sin(A-30^\circ) \sin(A+30^\circ)}$
 $= \frac{\cos 2A + \cos 60^\circ}{\cos 60^\circ - \cos 2A} = \frac{2 \cos 2A + 1}{1 - 2 \cos 2A}$
 $= \frac{2 + \sec 2A}{\sec 2A - 2}$.

$$\begin{aligned}
 (2) \quad \text{Second side} &= 2 \left\{ \frac{1 + \cos 2\alpha}{2} \cdot \frac{1 + \cos 2\beta}{2} + \frac{1 - \cos 2\alpha}{2} \cdot \frac{1 - \cos 2\beta}{2} \right\} \\
 &= \frac{1}{2} \{ 2 + 2 \cos 2\alpha \cos 2\beta \} = 1 + \cos 2\alpha \cos 2\beta.
 \end{aligned}$$

112. From a diagram it is easily seen that CD is equal to AC , and that from the right-angled triangle ABC ,

$$\begin{aligned}
 AC &= BC \cos 30^\circ = 132 \frac{\sqrt{3}}{2} \\
 &= 66 \times \frac{19}{11} = 114 \text{ yards.}
 \end{aligned}$$

Also the perp. from A on $BC = AC \sin 30^\circ = 57$ yards.

$$113. \quad \text{Since } \frac{a}{a+b+c} = \frac{\sin A}{\sin A + \sin B + \sin C}$$

$$\begin{aligned}
 &= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}, \quad [\text{XII. d. Ex. 3.}] \\
 &= \frac{1}{2} \left\{ \frac{\cos \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} + \dots - \dots \right\}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \frac{a \cot \frac{A}{2} + b \cot \frac{B}{2} - c \cot \frac{C}{2}}{a+b+c} &= \frac{1}{2} \left\{ \frac{\cos \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} + \dots - \dots \right\} \\
 &= \frac{1}{2} \left\{ \frac{\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} \right\}.
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} &= \frac{1}{2} \left\{ 2 \cos^2 \frac{A}{2} + 1 + \cos B - 1 - \cos C \right\} \\
 &= \frac{1}{2} \left\{ 2 \cos^2 \frac{A}{2} + 2 \sin \frac{B+C}{2} \sin \frac{C-B}{2} \right\} \\
 &= \cos \frac{A}{2} \left\{ \cos \frac{A}{2} + \sin \frac{C-B}{2} \right\} \\
 &= \cos \frac{A}{2} \left\{ \sin \frac{B+C}{2} + \sin \frac{C-B}{2} \right\} \\
 &= 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.
 \end{aligned}$$

Whence the required result easily follows.

$$114. \quad \text{Here} \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{40 \times 42}{75 \times 77}}$$

$$= \frac{2}{\sqrt{5}} = \frac{2\sqrt{2}}{\sqrt{10}},$$

$$\log \cos \frac{A}{2} = \frac{3}{2} \log 2 - \frac{1}{2} \log 10$$

$$= .4515450 - .5$$

$$= \bar{1}.9515450$$

$$\log \cos 26^\circ 34' = \frac{\bar{1}.9515389}{61}$$

$$\therefore \frac{A}{2} \text{ is less than } 26^\circ 34' \text{ by } \frac{61}{632} \times 60'';$$

$$\text{that is,} \quad \frac{A}{2} = 26^\circ 33' 54.2'',$$

$$\text{or} \quad A = 53^\circ 7' 48''.$$

$$115. \quad \sin (A - 90^\circ) = -\sin (90^\circ - A) = -\cos A$$

$$= -\sqrt{1 - \sin^2 A} = -\sqrt{6.4}$$

$$= -(\pm .8);$$

but A is between 90° and 180° , therefore $\sin (A - 90^\circ)$ is positive; that is

$$\sin (A - 90^\circ) = .8.$$

$$\operatorname{cosec} (270^\circ - A) = \operatorname{cosec} (180^\circ + 90^\circ - A)$$

$$= -\operatorname{cosec} (90^\circ - A)$$

$$= -\left(\frac{1}{\pm .8}\right)$$

$$= \pm 1.25;$$

but between the given limits $\operatorname{cosec} (270^\circ - A)$ must be positive, that is, the required value is 1.25.

116. By Art. 168,

$$\log_b c = \frac{\log_a c}{\log_a b}, \quad \log_c d = \frac{\log_a d}{\log_a c};$$

$$\begin{aligned} \text{hence the expression on the right} &= \log_a b \times \frac{\log_a c}{\log_a b} \times \frac{\log_a d}{\log_a c} \\ &= \log_a d. \end{aligned}$$

$$\log_{10} 2 = 1 - \log_{10} 5 = .30103;$$

$$\log_{10} 8 = 3 \log_{10} 2 = .90309.$$

$$\log_8 10 = \frac{1}{\log_{10} 8} = 1.1073093.$$

$$\begin{aligned} \log_{10} (.032)^5 &= 5 \log \frac{32}{1000} = 25 \log 2 - 15 \\ &= 7.52575 - 15 \\ &= \bar{8}.52575. \end{aligned}$$

$$\begin{aligned} 117. \quad \text{First side} &= \cos (360^\circ + 60^\circ + A) + \cos (60^\circ - A) \\ &= \cos (60^\circ + A) + \cos (60^\circ - A) \\ &= 2 \cos 60^\circ \cos A = \cos A. \end{aligned}$$

For the second part, put $A = 45^\circ$.

$$118. \quad \text{Write } t \text{ for } \tan \frac{x}{2}, \text{ then the equation may be written}$$

$$\frac{1-t^2}{1+t^2} - \sin \alpha \cot \beta \frac{2t}{1+t^2} = \cos \alpha;$$

$$\therefore t^2 (1 + \cos \alpha) + 2t \sin \alpha \cot \beta - (1 - \cos \alpha) = 0;$$

$$t^2 + 2 \cot \beta \frac{\sin \alpha}{1 + \cos \alpha} \cdot t - \frac{1 - \cos \alpha}{1 + \cos \alpha} = 0,$$

$$t^2 + 2 \cot \beta \tan \frac{\alpha}{2} t - \tan^2 \frac{\alpha}{2} = 0,$$

$$\left(t + \tan \frac{\alpha}{2} \cot \frac{\beta}{2} \right) \left(t - \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \right) = 0, \quad [\text{XI. d. Ex. 15.}]$$

$$\therefore \tan \frac{x}{2} = -\tan \frac{\alpha}{2} \cot \frac{\beta}{2}, \text{ or } \tan \frac{\alpha}{2} \tan \frac{\beta}{2}.$$

$$119. \quad \text{We have} \quad \frac{\sin (2A + B)}{\sin B} = \frac{m}{n};$$

$$\therefore \frac{m-n}{m+n} = \frac{\sin (2A + B) - \sin B}{\sin (2A + B) + \sin B}$$

$$= \frac{2 \cos (A + B) \sin A}{2 \sin (A + B) \cos A}$$

$$= \cot (A + B) \tan A.$$

120. Let x feet be the height of the tower ;
 then $\angle ABD = 90^\circ - \angle ADB$
 $= 45^\circ = \angle ADB ;$
 $\therefore AD = AB = x ;$
 $\therefore AC = x - 17.$

Now from $\triangle ABC,$

$$x^2 + (x - 17)^2 = 53^2 ;$$

$$\therefore x^2 - 17x - 1260 = 0,$$

$$(x - 45)(x + 28) = 0 ;$$

$$\therefore x = 45.$$

Again,

$$\tan ACB = \frac{45}{28} ;$$

but

$$\tan 31^\circ 48' = \cdot 62 = \frac{56}{90} = \frac{28}{45} ;$$

$$\therefore \angle ACB = 90^\circ - 31^\circ 48'$$

$$= 58^\circ 12'.$$

122. $\log 6 = \frac{1}{2} \log 36 = \cdot 778151,$
 $\log 8 = \log 48 - \log 6 = 1\cdot 681241 - \cdot 778151$
 $= \cdot 90309 ;$

$$\therefore \log 2 = \frac{1}{3} \log 8 = \cdot 30103 ;$$

$$\therefore \log 3 = \log 6 - \log 2 = \cdot 477121.$$

Now

$$\log 40 = 1 + 2 \log 2 = 1\cdot 60206.$$

$$\log \sqrt{\frac{2}{15}} = \frac{1}{2} (\log 4 - \log 30)$$

$$= \frac{1}{2} (2 \log 2 - \log 3 - 1)$$

$$= \bar{1}\cdot 562469, \text{ on substitution.}$$

123. We have $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{9}{10} \cot 72^\circ.$

$$\log \tan \frac{B-C}{2} = 2 \log 3 - 1 + \log \cot 72^\circ$$

$$= \bar{1}\cdot 4660186$$

$$\log \tan 16^\circ 18' = \frac{\bar{1}\cdot 4660078}{108}$$

$$\frac{108}{4687} \times 60'' = 1''.$$

$$\therefore \frac{B-C}{2} = 16^\circ 18' 1'', \quad \frac{B+C}{2} = 18^\circ ;$$

$$\therefore B = 34^\circ 18' 1'', \quad C = 1^\circ 41' 59''.$$

$$124. \quad (1) \quad \cos A + \cos B \cos C = -\cos B \overline{\cos C} + \cos B \cos C \\ = \sin B \sin C;$$

$$\therefore \text{Second side} = a^2 \sin B \sin C = b \sin A \cdot c \sin A \\ = bc \sin^2 A.$$

$$(2) \quad \text{First side} = c(b \cos A + a \cos B) + 2ab \cos C \\ = c^2 + 2ab \cos C = a^2 + b^2.$$

$$125. \quad \tan \frac{\beta - \alpha}{2} = \frac{\tan \frac{\beta}{2} - \tan \frac{\alpha}{2}}{1 + \tan \frac{\beta}{2} \tan \frac{\alpha}{2}} = \frac{3 \tan \frac{\alpha}{2}}{1 + 4 \tan^2 \frac{\alpha}{2}} \\ = \frac{3 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2} + 4 \sin^2 \frac{\alpha}{2}} = \frac{3 \sin \alpha}{2 \cos^2 \frac{\alpha}{2} + 8 \sin^2 \frac{\alpha}{2}} \\ = \frac{3 \sin \alpha}{1 + \cos \alpha + 4(1 - \cos \alpha)} = \frac{3 \sin \alpha}{5 - 3 \cos \alpha}.$$

$$126. \quad \sin(36^\circ + A) - \sin(36^\circ - A) = 2 \cos 36^\circ \sin A = \frac{\sqrt{5} + 1}{2} \sin A;$$

$$\sin(72^\circ - A) - \sin(72^\circ + A) = 2 \cos 72^\circ \sin(-A) = -\frac{\sqrt{5} - 1}{2} \sin A.$$

By addition we obtain the required result.

$$127. \quad \tan \theta = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} = \frac{-\frac{2}{3}}{\pm \sqrt{\frac{5}{9}}} = \pm \frac{2}{\sqrt{5}}.$$

The boundary line of θ is in the 3rd or 4th quadrant, hence the tangent is positive in one case and negative in the other.

$$128. \quad (1) \quad \text{First side} = \sin 2A + \sin \left(\frac{\pi}{2} - 2B \right) \\ = 2 \sin \left(\frac{\pi}{4} + A - B \right) \cos \left(\frac{\pi}{4} - A - B \right).$$

$$(2) \quad \text{First side} = 2 \sin \frac{\theta - \phi}{2} \cos \frac{\theta + \phi}{2} \cdot 2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} \\ = 2 \sin(\theta - \phi) \cos^2 \frac{\theta + \phi}{2}.$$

129. Since $a : b : c = \sin A : \sin B : \sin C$, we have

$$(a + b + c)(a + b - c) = 3ab,$$

or

$$(a + b)^2 - c^2 = 3ab,$$

that is, $\frac{a^2 + b^2 - c^2}{2ab} = \frac{1}{2}$; $\therefore \cos C = \frac{1}{2}$, and $C = 60^\circ$.

130. Let $a^x = b$, $a^y = d$, $c^p = d$, $c^q = b$,

then we have to prove $px = qy$.

Now $a^x = c^q$, and $a^y = c^p$;

$$\therefore a^{px} = c^{pq} = a^{qy}; \text{ that is } px = qy.$$

131. $\log 20 \cdot 01 = 1 \cdot 3012471$

$$\log 20 \cdot 00 = 1 \cdot 3010300$$

$$\text{diff. for } \cdot 01 = \frac{2171}{2171}$$

$$\therefore \text{diff. for } \cdot 0075 = \frac{3}{4} \times 2171 = 1628;$$

$$\therefore \log 20 \cdot 0075 = 1 \cdot 3011928.$$

132. Let AD be the median from B ; then

$$AB^2 + BC^2 = 2(AD^2 + BD^2);$$

that is,

$$49 + 81 = 2x^2 + 32;$$

whence

$$x = 7.$$

133. We have

$$\frac{1 + \sin A}{\cos A} = 2,$$

$$\therefore (1 + \sin A)^2 = 4(1 - \sin^2 A),$$

rejecting the factor $1 + \sin A$, which gives an inadmissible value, we have

$$1 + \sin A = 4(1 - \sin A);$$

whence

$$\sin A = \frac{3}{5}.$$

$$134. \text{ First side} = \frac{4(1 - \cos 2A) - (1 - \cos 4A)}{4(1 + \cos 2A) - (1 - \cos 4A)}$$

$$= \frac{8 \sin^2 A - 2 \sin^2 2A}{8 \cos^2 A - 2 \sin^2 2A}$$

$$= \frac{\sin^2 A (1 - \cos^2 A)}{\cos^2 A (1 - \sin^2 A)} = \tan^4 A.$$

$$\begin{aligned}
 135. \quad \text{First side} &= \frac{3 \sin A - 4 \sin^3 A + 4 \cos^3 A - 3 \cos A}{3 \sin A - 4 \sin^3 A - (4 \cos^3 A - 3 \cos A)} \\
 &= \frac{3(\sin A - \cos A) - 4(\sin^3 A - \cos^3 A)}{3(\sin A + \cos A) - 4(\sin^3 A + \cos^3 A)} \\
 &= \frac{\sin A - \cos A}{\sin A + \cos A} \cdot \frac{3 - 4(\sin^2 A + \cos^2 A + \sin A \cos A)}{3 - 4(\sin^2 A + \cos^2 A - \sin A \cos A)} \\
 &= \frac{\tan A - 1}{\tan A + 1} \cdot \frac{3 - 4(1 + \sin A \cos A)}{3 - 4(1 - \sin A \cos A)} \\
 &= \frac{\tan A - 1}{1 + \tan A} \cdot \frac{1 + 2 \sin 2A}{1 - 2 \sin 2A} = \frac{1 + 2 \sin 2A}{1 - 2 \sin 2A} \tan(A - 45^\circ).
 \end{aligned}$$

$$136. \quad \text{Put } \frac{x}{\cos A} = \frac{y}{\cos B} = k; \text{ then}$$

$$\begin{aligned}
 \text{First side} &= k \cos A \tan A + k \cos B \tan B \\
 &= k(\sin A + \sin B) \\
 &= 2k \sin \frac{A+B}{2} \cos \frac{A-B}{2} \\
 &= 2k \cos \frac{A+B}{2} \cos \frac{A-B}{2} \tan \frac{A+B}{2} \\
 &= k(\cos A + \cos B) \tan \frac{A+B}{2} \\
 &= (x+y) \tan \frac{A+B}{2}.
 \end{aligned}$$

$$\begin{aligned}
 137. \quad \log 7 &= \log 24.5 - \log 3.5 = .845098; \\
 \log 5 &= \log 35 - \log 7 = .69897; \\
 \log 13 &= \log 3.25 - \log .25 \\
 &= \log 3.25 - (2 \log 5 - 2) = 1.113943.
 \end{aligned}$$

$$138. \quad \text{Here } \tan A = \frac{a}{b} = \frac{384}{330} = \frac{128}{110}.$$

$$\log 128 = 7 \log 2 = 2.1072100$$

$$\log 110 = 2.0413927$$

$$\log \tan A = .0658173$$

$$\log \tan 49^\circ 19' = .0656886$$

$$\text{diff.} \qquad \qquad \qquad 1287$$

$$\therefore \text{ prop}^1 \text{ increase} = \frac{1287}{2555} \times 60'' = 30'';$$

$$\therefore A = 49^\circ 19' 30''; B = 40^\circ 40' 30''.$$

$$\begin{aligned}
 139. \quad \text{First side} &= \frac{2 \sin \frac{\theta + \alpha}{2} \sin \frac{\theta - \alpha}{2}}{2 \cos \frac{\theta + \alpha}{2} \cos \frac{\theta - \alpha}{2}} = \frac{\cos \alpha - \cos \theta}{\cos \alpha + \cos \theta} \\
 &= \frac{\cos \alpha - \cos \alpha \cos \beta}{\cos \alpha + \cos \alpha \cos \beta} = \frac{1 - \cos \beta}{1 + \cos \beta} \\
 &= \tan^2 \frac{\beta}{2}.
 \end{aligned}$$

140. We have

$$\begin{aligned}
 \frac{\sin \theta}{\cos \theta - \sin \phi} - \frac{\sin \theta}{\cos \theta + \sin \phi} &= \frac{2 \sin \theta \sin \phi}{\cos^2 \theta - \sin^2 \phi} \\
 &= \frac{\sin \phi \{ \cos \phi + \sin \theta - (\cos \phi - \sin \theta) \}}{1 - \sin^2 \theta - (1 - \cos^2 \phi)} \\
 &= \frac{\sin \phi \{ \cos \phi + \sin \theta - (\cos \phi - \sin \theta) \}}{(\cos \phi + \sin \theta) (\cos \phi - \sin \theta)} \\
 &= \frac{\sin \phi}{\cos \phi - \sin \theta} - \frac{\sin \phi}{\cos \phi + \sin \theta}.
 \end{aligned}$$

$$141. \quad \text{We have } \frac{c^2 (a+b)^2 s (s-b)}{ac} = \frac{b^2 (a+c)^2 s (s-c)}{ab};$$

$$c (a^2 + b^2 + 2ab) (s-b) = b (a^2 + c^2 + 2ac) (s-c);$$

$$\text{that is, } a^2 s (c-b) - bcs (c-b) + bc (c^2 - b^2) + 2abc (c-b) = 0,$$

$$\text{or } (c-b) \{ a^2 s - bc (s-c-b) + 2abc \} = 0,$$

$$\text{or } (c-b) \{ a^2 s - bc (a-s) + 2abc \} = 0,$$

$$\text{or } (c-b) \{ a^2 s + bcs + 3abc \} = 0;$$

therefore $b-c=0$, since the other factor evidently cannot be zero.

$$142. \quad (1) \quad \cot A + \operatorname{cosec} A = \frac{1 + \cos A}{\sin A} = \cot \frac{A}{2}; \quad [\text{XI. d. Ex. 11}]$$

$$\begin{aligned}
 \tan A + \sec A &= \frac{1 + \sin A}{\cos A} = \frac{1 - \cos \left(\frac{\pi}{2} + A \right)}{\sin \left(\frac{\pi}{2} + A \right)} \\
 &= \tan \left(\frac{\pi}{4} + \frac{A}{2} \right);
 \end{aligned}$$

therefore by division the required result is obtained.

(2) Since $\sin 3A = 3 \sin A - 4 \sin^3 A$, we have

$$\begin{aligned} \text{First side} &= \frac{1}{4} \{ (3 \sin A - \sin 3A) + \dots + \dots \} \\ &= \frac{3}{4} \{ \sin A + \sin (120^\circ + A) + \sin (240^\circ + A) \} \\ &\quad - \frac{1}{4} \{ \sin 3A + \sin (360^\circ + 3A) + \sin (720^\circ + 3A) \} \\ &= -\frac{3}{4} \sin 3A. \quad [\text{See solution of XII. c. 25.}] \end{aligned}$$

143. Let

$$x = \sqrt[5]{18 \times .0015},$$

then

$$x = \sqrt[5]{.027} = (.3)^{\frac{3}{5}},$$

$$\frac{3}{5} \log .3 = \frac{3}{5} (\bar{1}.4771213)$$

$$= \bar{1}.6862728$$

diff. for .00001 = 31 ;

$$\log .48559 = \frac{\bar{1}.6862697}{31}$$

$$\therefore \text{prop}^1 \text{ increase} = \frac{31}{90} \times .00001$$

$$= .000003 ;$$

$$\therefore x = .485593.$$

144.

$$\sin B = \frac{b \sin A}{a} = \frac{394}{573} \cos 22^\circ 4'.$$

$$\log 394 = 2.5954962$$

$$\log \cos 22^\circ 4' = \bar{1}.9669614$$

$$2.5624576$$

$$\log 573 = 2.7581546$$

$$\log \sin B = \bar{1}.8043030$$

$$\log \sin 39^\circ 35' = \bar{1}.8042757$$

$$\text{diff.} \quad 273$$

$$\text{diff. for } 60'' = 1527 ;$$

$$\therefore \text{prop}^1 \text{ increase} = \frac{273}{1527} \times 60'' = 10.7'' ;$$

$$\therefore B = 39^\circ 35' 11'' ; \text{ and } C = 28^\circ 20' 49''.$$

145. First side = $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$

$$= \frac{b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2}{2abc}$$

$$= \frac{a^2 + b^2 + c^2}{2abc}.$$

146.

$$\log 119 = \log 7 + \log 17 = 2.0755469,$$

$$\log \frac{17}{7} = \log 17 - \log 7 = .3853509,$$

$$\begin{aligned} \log \frac{289}{343} &= \log 17^2 - \log 7^3 \\ &= 2 \log 17 - 3 \log 7 = 1.9256038. \end{aligned}$$

147. We have

$$\begin{aligned} \cos \theta &= \frac{\sin A}{\sin B + \sin C} \\ &= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}} \\ &= \frac{\sin \frac{A}{2}}{\cos \frac{B-C}{2}} = \frac{\cos \frac{B+C}{2}}{\cos \frac{B-C}{2}}; \\ \therefore \tan^2 \frac{\theta}{2} &= \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{\cos \frac{B-C}{2} - \cos \frac{B+C}{2}}{\cos \frac{B-C}{2} + \cos \frac{B+C}{2}} \\ &= \frac{2 \sin \frac{B}{2} \sin \frac{C}{2}}{2 \cos \frac{B}{2} \cos \frac{C}{2}} = \tan \frac{B}{2} \tan \frac{C}{2}. \end{aligned}$$

148. Let r be the radius of the circle, x, y the side of circumscribing equilateral triangle and hexagon respectively.

Then from the figure of Art. 215,

$$x = 2r \tan 60^\circ = 2r\sqrt{3}; \quad y = 2r \tan 30^\circ = \frac{2r}{\sqrt{3}};$$

whence

$$xy = 4r^2 = (2r)^2.$$

149. From the equation $a^2 = b^2 + c^2 - 2bc \cos A$, we have on substitution and reduction

$$c^2 - 150\sqrt{2} \cdot c + 10000 = 0;$$

$$\therefore (c - 100\sqrt{2})(c - 50\sqrt{2}) = 0;$$

$$\therefore c = 100\sqrt{2}, \text{ or } 50\sqrt{2}.$$

Again

$$\sin B = \frac{b \sin A}{a} = \frac{150}{50\sqrt{10}} = \frac{3}{\sqrt{10}},$$

$$\log \sin B = \log 3 - \frac{1}{2} \log 10$$

$$= \bar{1}.9771213,$$

$$\log \sin 71^\circ 33' = \bar{1}.9770832$$

$$\text{diff.} \quad \quad \quad 381$$

$$\text{diff. for } 60'' = 421;$$

$$\therefore \text{prop}^1 \text{ increase} = \frac{381}{421} \times 60'' = 54'';$$

$$\therefore B = 71^\circ 33' 54'', \text{ or } 180^\circ - (71^\circ 33' 54'').$$

150. The series may be written

$$(\operatorname{cosec} x - \operatorname{cosec} 3x) + (\operatorname{cosec} 3x - \operatorname{cosec} 3^2 x) + \dots + (\operatorname{cosec} 3^{n-1} x - \operatorname{cosec} 3^n x),$$

which reduces to

$$\operatorname{cosec} x - \operatorname{cosec} 3^n x.$$

153. First side

$$= \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} - \frac{3 \cos \frac{3\theta}{2}}{\sin \frac{3\theta}{2}}$$

$$= \frac{\sin \frac{3\theta}{2} \cos \frac{\theta}{2} - 3 \sin \frac{\theta}{2} \cos \frac{3\theta}{2}}{\sin \frac{\theta}{2} \sin \frac{3\theta}{2}}$$

$$= \frac{\sin \left(\frac{3\theta}{2} - \frac{\theta}{2} \right) - 2 \sin \frac{\theta}{2} \cos \frac{3\theta}{2}}{\sin \frac{\theta}{2} \sin \frac{3\theta}{2}}$$

$$= \frac{\sin \theta - (\sin 2\theta - \sin \theta)}{\frac{1}{2} (\cos \theta - \cos 2\theta)}$$

$$= \frac{4 \sin \theta - 2 \sin 2\theta}{1 + \cos \theta - 2 \cos^2 \theta} = \frac{4 \sin \theta (1 - \cos \theta)}{(1 + 2 \cos \theta) (1 - \cos \theta)}$$

$$= \frac{4 \sin \theta}{1 + 2 \cos \theta}.$$

154. We have, by Art. 135, Ex. 2,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C;$$

$$\therefore \frac{3}{4} + \frac{5}{12} + \tan C = \frac{3}{4} \cdot \frac{5}{12} \cdot \tan C;$$

whence $\tan C = -\frac{56}{33}.$

Also $\cos A = \frac{1}{\sqrt{1+\tan^2 A}} = \frac{1}{\sqrt{1+\frac{9}{16}}} = \frac{4}{5},$

$$\cos B = \frac{1}{\sqrt{1+\tan^2 B}} = \frac{1}{\sqrt{1+\frac{25}{144}}} = \frac{12}{13},$$

$$\cos C = \frac{1}{\sqrt{1+\tan^2 C}} = \frac{1}{\sqrt{1+\frac{3136}{1089}}} = -\frac{33}{65},$$

the negative sign of the radical being taken in the third case since C is an obtuse angle.

Again $\tan C = -\frac{56}{33}; \therefore \tan (180^\circ - C) = \frac{56}{33},$

$$\log 56 = 1.7481880$$

$$\log 33 = 1.5185139$$

$$\log \tan (180^\circ - C) = .2296741$$

$$\log \tan 59^\circ 29' = .2295627$$

$$\text{diff.} \quad 1114$$

$$\therefore \text{prop}^1 \text{ increase} = \frac{1114}{2888} \times 60'' = 23'';$$

$$\therefore 180^\circ - C = 59^\circ 29' 23'';$$

that is,

$$C = 120^\circ 30' 37''.$$

155. We have

$$\frac{\sin (A-B)}{\sin (A+B)} = \frac{a^2 - b^2}{a^2 + b^2}$$

$$= \frac{\sin^2 A - \sin^2 B}{\sin^2 A + \sin^2 B};$$

$$\therefore \frac{\sin (A-B)}{\sin C} = \frac{\sin (A-B) \sin (A+B)}{\sin^2 A + \sin^2 B};$$

$\therefore$ either $\sin (A-B) = 0 \dots\dots\dots (1),$

or $\sin^2 C = \sin^2 A + \sin^2 B \dots\dots\dots (2).$

If (1) is true, $A=B$; if (2) is true, we have $c^2 = a^2 + b^2.$

$$\begin{aligned}
 156. \quad \text{The first side} &= \frac{2\Delta}{s} \cdot \frac{abc}{\Delta} \left\{ \frac{s(s-a)}{bc} + \dots + \dots \right\} \\
 &= 2 \{ a(s-a) + b(s-b) + c(s-c) \} \\
 &= (a+b+c)^2 - 2a^2 - 2b^2 - 2c^2 \\
 &= 2bc + 2ca + 2ab - a^2 - b^2 - c^2.
 \end{aligned}$$

157. We have

$$\begin{aligned}
 \frac{\cos C}{\sin B \cos A} - \frac{\cos B}{\sin C \cos A} &= \frac{\cos(A+C)}{\sin C \cos A} - \frac{\cos(A+B)}{\sin B \cos A} \\
 &= \cot C - \tan A - (\cot B - \tan A) \\
 &= \cot C - \cot B.
 \end{aligned}$$

$$\begin{aligned}
 158. \quad \log 3 &= \log 18 - \log 6; \quad \log 2 = \log 6 - \log 3; \\
 \log 11 &= \log 44 - \log 4 = \log 44 - 2 \log 2.
 \end{aligned}$$

159. (1) We have

$$\begin{aligned}
 &\tan(60^\circ + A) \tan(60^\circ - A) \\
 &= \frac{2 \sin(60^\circ + A) \sin(60^\circ - A)}{2 \cos(60^\circ + A) \cos(60^\circ - A)} = \frac{\cos 2A - \cos 120^\circ}{\cos 2A + \cos 120^\circ} = \frac{2 \cos 2A + 1}{2 \cos 2A - 1}; \\
 \therefore \text{second side} &= 2 \sin A \cos 2A + \sin A \\
 &= (\sin 3A - \sin A) + \sin A = \sin 3A.
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \text{First side} &= 2 \sin(A-B) \cos(A+B) \frac{\sin(A+B)}{\cos(A+B)} \\
 &= 2 \sin(A+B) \sin(A-B) \\
 &= 2 \sin^2 A - \sin^2 B. \quad [\text{Art. 114.}]
 \end{aligned}$$

$$\begin{aligned}
 160. \quad \tan \frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{113 \times 101}{296 \times 82}} \\
 \log 296 &= 2.4712917 & \log 113 &= 2.0530784 \\
 \log 82 &= 1.9138139 & \log 101 &= 2.0043214 \\
 &4.3851056 & &4.0573998 \\
 & & &4.3851056 \\
 & & &2 \overline{1.6722942} \\
 \log \tan \frac{C}{2} &= \overline{1.8361471} \\
 \log \tan 34^\circ 26' &= \overline{1.8360513} \\
 \text{diff.} & & &958 \\
 \text{prop}^l. \text{ increase} &= \frac{958}{2708} \times 60'' = 21''; \\
 \therefore C &= 68^\circ 52' 42''.
 \end{aligned}$$

161. Let a be a side of the octagon, r the radius of the circle, then $8a = 2\pi r$; and we have

$$\begin{aligned}\frac{\text{area of circle}}{\text{area of octagon}} &= \frac{\pi r^2}{2a^2 \cot \frac{\pi}{8}} = \frac{8\pi r^2}{\pi^2 r^2} \tan \frac{\pi}{8} \\ &= \frac{8(\sqrt{2}-1)}{\pi} = \frac{.414 \times 8}{3.1416} = \frac{4140}{3927} = \frac{1380}{1309}.\end{aligned}$$

162. We have $2b = a + c$, or $a = 2b - c$;

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + c^2 - (2b - c)^2}{2bc} = \frac{4bc - 3b^2}{2bc} = \frac{4c - 3b}{2c}.$$

163. We have $a(\sin \theta \cos \alpha + \cos \theta \sin \alpha) = b(\sin \theta \cos \beta + \cos \theta \sin \beta)$;

$$\therefore \sin \theta (a \cos \alpha - b \cos \beta) = \cos \theta (b \sin \beta - a \sin \alpha);$$

that is,
$$\cot \theta = \frac{a \cos \alpha - b \cos \beta}{b \sin \beta - a \sin \alpha}.$$

164. Let a, b, A be the given parts; then $R = \frac{a}{2 \sin A}$, which is the same for each triangle.

165. Let NS be a horizontal line pointing North and South. Then if K be the position of the kite, and KD is the vertical from K , we have S, B, D, A, N in a straight line, and $BA = c$. Also $KA = \frac{c \sin \beta}{\sin BKA} = \frac{c \sin \beta}{\sin (\alpha + \beta)}$.

And $KD = KA \sin \alpha = \frac{c \sin \alpha \sin \beta}{\sin (\alpha + \beta)}.$

$$\begin{aligned}166. \quad \cos \alpha + \cos \beta + \cos \gamma + 1 &= 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} + 2 \cos^2 \frac{\gamma}{2} \\ &= 2 \cos \left(\pi - \frac{\gamma}{2} \right) \cos \frac{\alpha - \beta}{2} + 2 \cos^2 \frac{\gamma}{2} \\ &= -2 \cos \frac{\gamma}{2} \cos \frac{\alpha - \beta}{2} + 2 \cos^2 \frac{\gamma}{2} \\ &= 2 \cos \frac{\gamma}{2} \left\{ \cos \frac{\gamma}{2} - \cos \frac{\alpha - \beta}{2} \right\} \\ &= 2 \cos \frac{\gamma}{2} \left\{ \cos \left(\pi - \frac{\alpha + \beta}{2} \right) - \cos \frac{\alpha - \beta}{2} \right\} \\ &= -2 \cos \frac{\gamma}{2} \left\{ \cos \frac{\alpha + \beta}{2} + \cos \frac{\alpha - \beta}{2} \right\} \\ &= -4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}.\end{aligned}$$

$$167. \text{ First side} = 2 \cos 9A \cos A + 6 \cos 3A \cos A$$

$$= 2 \cos A \{ \cos 9A + 3 \cos 3A \}$$

$$= 2 \cos A \{ (4 \cos^3 3A - 3 \cos 3A) + 3 \cos 3A \}$$

$$= 8 \cos A \cos^3 3A.$$

$$168. \quad r_1 + r_2 = \frac{\Delta}{s-a} + \frac{\Delta}{s-b} = \frac{\Delta (s-a+s-b)}{(s-a)(s-b)} = c \cdot \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}.$$

$$\text{Also} \quad r_2 r_3 + r_3 r_1 + r_1 r_2 = s^2 \quad [\text{XVIII. a. Ex. 24}];$$

$$\begin{aligned} \therefore \text{ First side} &= \frac{abc}{4s^2} \sqrt{\frac{s(s-c)}{(s-a)(s-b)} \cdot \frac{s(s-a)}{(s-b)(s-c)} \cdot \frac{s(s-b)}{(s-c)(s-a)}} \\ &= \frac{abc}{4 \sqrt{s(s-a)(s-b)(s-c)}} = \frac{abc}{4\Delta} = R. \end{aligned}$$

$$\begin{aligned} 169. \quad \cos 2\theta &= \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = -\frac{2 \tan^2 \phi}{2(1 + \tan^2 \phi)} \\ &= -\sin^2 \phi. \end{aligned}$$

$$\begin{aligned} 170. \quad \text{First side} &= \frac{\sin A \sin (60^\circ + A) \sin (120^\circ + A)}{\cos A \cos (60^\circ + A) \cos (120^\circ + A)} \\ &= \frac{\sin A}{\cos A} \cdot \frac{\cos 60^\circ - \cos (180^\circ + 2A)}{\cos 60^\circ + \cos (180^\circ + 2A)} \\ &= \frac{\sin A}{\cos A} \cdot \frac{1 + 2 \cos 2A}{1 - 2 \cos 2A} \\ &= \frac{\sin A + 2 \sin A \cos 2A}{\cos A - 2 \cos A \cos 2A} \\ &= \frac{\sin A + (\sin 3A - \sin A)}{\cos A - (\cos 3A + \cos A)} \\ &= -\frac{\sin 3A}{\cos 3A} = -\tan 3A. \end{aligned}$$

$$171. \quad \text{We have } A - B = B; \text{ therefore } \sin (A - B) = \sin B;$$

$$\text{also} \quad \sin (A + B) = \sin (180^\circ - C) = \sin C;$$

$$\therefore \sin (A + B) \sin (A - B) = \sin C \sin B;$$

$$\text{that is,} \quad \sin^2 A - \sin^2 B = \sin B \sin C;$$

$$\text{or} \quad a^2 - b^2 = bc.$$

172. If a, b be the sides of the triangle and square respectively, and R the radius of the circle, it is easy to shew that

$$a = R\sqrt{3}; \quad b = 2R \cos 45^\circ = R\sqrt{2}.$$

$$\begin{aligned} 173. \quad \text{We have } \frac{c+b}{c-b} \tan \frac{A}{2} &= \frac{\sin C + \sin B}{\sin C - \sin B} \tan \frac{A}{2} \\ &= \frac{\sin(A+B) + \sin B}{\sin(A+B) - \sin B} \tan \frac{A}{2} \\ &= \frac{2 \sin\left(\frac{A}{2} + B\right) \cos \frac{A}{2}}{2 \cos\left(\frac{A}{2} + B\right) \sin \frac{A}{2}} \tan \frac{A}{2} \\ &= \tan\left(\frac{A}{2} + B\right). \end{aligned}$$

$$\tan\left(\frac{A}{2} + B\right) = \frac{7b+3b}{7b-3b} \tan \frac{A}{2} = \frac{10}{4} \tan \frac{A}{2},$$

$$\log 10 = 1$$

$$\log 4 = \cdot 60206$$

$$\cdot 3979400$$

$$\log \tan 3^\circ 18' 42'' = \bar{2} \cdot 7624069$$

$$\log \tan\left(\frac{A}{2} + B\right) = \bar{1} \cdot 1603469$$

$$\log \tan 8^\circ 13' 50'' = \bar{1} \cdot 1603083$$

$$386$$

$$\begin{aligned} \text{prop}^1 \text{ increase} &= \frac{386}{1486} \times 60'' \\ &= 2 \cdot 6''. \end{aligned}$$

$$\therefore \frac{A}{2} + B = 8^\circ 13' 53'', \text{ and } \frac{A}{2} = 3^\circ 18' 42'';$$

$$\therefore B = 4^\circ 55' 11'',$$

$$C = 168^\circ 27' 25''.$$

174. By a well-known geometrical property, we have

$$AC^2 + AB^2 = 2AD^2 + 2DB^2.$$

$$\therefore AC^2 - AB^2 = 2(AD^2 + DB^2 - AB^2)$$

$$= 4AD \cdot DB \cos ADB$$

$$= 4AD \cdot DB \sin ADB \cot ADB$$

$$= (AD \cdot DB \sin ADB) 4 \cot ADB$$

$$= 4\Delta \cot ADB,$$

for $AD \cdot DB \sin ADB = 2(\text{area of triangle } ADB) = \Delta.$

$$\begin{aligned}
 175. \quad \tan 40^\circ \tan 80^\circ &= \frac{2 \sin 40^\circ \sin 80^\circ}{2 \cos 40^\circ \cos 80^\circ} \\
 &= \frac{\cos 40^\circ - \cos 120^\circ}{\cos 40^\circ + \cos 120^\circ} \\
 &= \frac{2 \cos 40^\circ + 1}{2 \cos 40^\circ - 1};
 \end{aligned}$$

$$\begin{aligned}
 \therefore \tan 20^\circ \tan 40^\circ \tan 80^\circ &= \frac{2 \sin 20^\circ \cos 40^\circ + \sin 20^\circ}{2 \cos 20^\circ \cos 40^\circ - \cos 20^\circ} \\
 &= \frac{\sin 60^\circ}{\cos 60^\circ} = \tan 60^\circ.
 \end{aligned}$$

176. See Art. 144.

$$\begin{aligned}
 177. \quad 2R &= \frac{abc}{2\Delta}; \quad 2r = \frac{2\Delta}{s}; \\
 \therefore 2R \cdot 2r &= \frac{abc}{s} = \frac{2abc}{a+b+c}.
 \end{aligned}$$

$$178. \quad \sin B = \frac{b}{a} \sin A = \frac{4\sqrt{3}}{7}$$

$$\frac{1}{2} \log 3 = \cdot 2385606$$

$$2 \log 2 = \cdot 6020600$$

$$\cdot 8406206$$

$$\log 7 = \cdot 8450980$$

$$\log \sin B = \bar{1} \cdot 9955226$$

$$\log \sin 81^\circ 47' = \bar{1} \cdot 9955188$$

$$\cdot 38$$

$$\begin{aligned}
 \text{prop}^l. \text{ increase} &= \frac{38}{183} \times 60'' \\
 &= 12 \cdot 4''.
 \end{aligned}$$

$\therefore B = 81^\circ 47' 12''$; but since $a < b$ there is another value of B supplementary to this, viz. $98^\circ 12' 48''$.

$$\therefore C = 68^\circ 12' 48'', \text{ or } 51^\circ 47' 12''.$$

To find c , we have $c^2 - 2b \cos A \cdot c + b^2 - a^2 = 0$, [Art. 150]

$$\text{that is} \quad c^2 - 24c + 143 = 0,$$

$$(c - 13)(c - 11) = 0;$$

$$\therefore c = 13, \text{ or } 11.$$

$$179. \quad \tan^2\left(\frac{\pi}{4} + \beta\right) = \frac{1 + \sin 2\beta}{1 - \sin 2\beta} \quad [\text{XI. f. Ex. 15}]$$

$$\begin{aligned} &= \frac{1 + \sin 2\alpha \sin 2\alpha' + \sin 2\alpha + \sin 2\alpha'}{1 + \sin 2\alpha \sin 2\alpha' - \sin 2\alpha - \sin 2\alpha'} \\ &= \frac{(1 + \sin 2\alpha)(1 + \sin 2\alpha')}{(1 - \sin 2\alpha)(1 - \sin 2\alpha')} \\ &= \tan^2\left(\frac{\pi}{4} + \alpha\right) \tan^2\left(\frac{\pi}{4} + \alpha'\right). \end{aligned}$$

180. With the figure of Art. 199, let

$$PC = x, \beta = 28^\circ, \alpha = 16^\circ, a = 16071 \text{ feet.}$$

Then

$$x = \frac{16071 \sin 28^\circ \sin 16^\circ}{\sin 12^\circ}.$$

$$\log 16071 = 4.2060$$

$$\log \sin 28^\circ = 1.6716$$

$$\log \sin 16^\circ = 1.4403$$

$$3.3179$$

$$\log \sin 12^\circ = 1.3179$$

$$\log x = 4.0000$$

$$\therefore x = 10000 \text{ feet.}$$

$$182. \quad \text{We have } 2 \cos \frac{A+C}{2} \cos \frac{A-C}{2} = 2 \sin (A+C)$$

$$= 4 \sin \frac{A+C}{2} \cos \frac{A+C}{2};$$

$$\therefore \text{either} \quad \cos \frac{A+C}{2} = 0 \dots\dots\dots(1),$$

$$\text{or} \quad \cos \frac{A-C}{2} = 2 \sin \frac{A+C}{2} \dots\dots\dots(2).$$

From (1) $\frac{A+C}{2} = (2n+1)\frac{\pi}{2}$, and from (2) by expanding each side and dividing throughout by $\cos \frac{A}{2} \cos \frac{C}{2}$ we obtain the other result.

$$183. \quad \text{First side} = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} - 2 \sin \frac{C}{2} \cos \frac{C}{2}$$

$$= 2 \sin \frac{C}{2} \left\{ \cos \frac{A-B}{2} - \cos \frac{C}{2} \right\}$$

$$\begin{aligned}
 &= 2 \sin \frac{C}{2} \left\{ \cos \frac{A+B}{2} - \cos \frac{\pi - (A+B)}{2} \right\} \\
 &= 2 \sin \frac{C}{2} \cdot 2 \sin \left(\frac{\pi}{4} - \frac{B}{2} \right) \sin \left(\frac{\pi}{4} - \frac{A}{2} \right) \\
 &= 4 \sin \frac{C}{2} \sin \left(45^\circ - \frac{A}{2} \right) \sin \left(45^\circ - \frac{B}{2} \right).
 \end{aligned}$$

$$184. \quad \frac{bc}{r_1} = \frac{bc(s-a)}{\Delta} = \frac{4R(s-a)}{a} = 2R \left(\frac{b+c-a}{a} \right);$$

$$\begin{aligned}
 \therefore \text{first side} &= 2R \left(\frac{b+c-a}{a} + \frac{c+a-b}{b} + \frac{a+b-c}{c} \right) \\
 &= 2R \left(\frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} + \frac{a}{c} + \frac{b}{c} - 3 \right).
 \end{aligned}$$

185. Since the points A, B, E, C are concyclic, $\angle BED = \angle C$; also

$$\angle EBD = \angle EAC = \frac{A}{2};$$

$$\therefore \text{from } \triangle BDE, \quad BD = \frac{DE \sin C}{\sin \frac{A}{2}},$$

$$\text{from } \triangle DEC, \quad DC = \frac{DE \sin B}{\sin \frac{A}{2}};$$

$$\therefore \text{by addition } a = \frac{DE}{\sin \frac{A}{2}} (\sin B + \sin C);$$

$$\therefore a^2 = \frac{DE^2}{\sin^2 \frac{A}{2}} (a \sin B + a \sin C)$$

$$= \frac{DE^2 (b+c) \sin A}{\sin^2 \frac{A}{2}};$$

$$\therefore DE = \frac{a^2 \sec \frac{A}{2}}{2(b+c)}.$$

$$186. \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{5875 \cdot 5 \times 1785 \cdot 5}{3850 \times 3811}}$$

$\log 5875 \cdot 5 = 3 \cdot 7690448$ $\log 1785 \cdot 5 = 3 \cdot 2517599$ $\quad 7 \cdot 0208047$ $\quad 7 \cdot 1664996$ $\quad 2 \overline{1 \cdot 8543051}$	$\log 3850 = 3 \cdot 5854607$ $\log 3811 = 3 \cdot 5810389$ $\quad 7 \cdot 1664996$
--	---

$$\log \cos \frac{A}{2} = \overline{1 \cdot 9271525}$$

$$\log \cos 32^\circ 16' = \overline{1 \cdot 9271509} \quad \text{prop}^l. \text{ decrease} = \frac{16}{797} \times 60''$$

$$\quad \quad \quad 16 \quad \quad \quad = 1 \cdot 2'';$$

$$\therefore \frac{A}{2} = 32^\circ 15' 58 \cdot 8'', \text{ or } A = 64^\circ 31' 58''.$$

188. See Art. 117.

$$190. \quad (1) \quad \frac{b^2 + c^2 - a^2}{a^2 + c^2 - b^2} = \frac{2bc \cos A}{2ca \cos B} = \frac{b \cos A}{a \cos B}$$

$$= \frac{\sin B \cos A}{\sin A \cos B} = \frac{\tan B}{\tan A}.$$

$$(2) \quad \frac{2 \sin^2 A}{a^2} = \frac{2 \sin^2 B}{b^2};$$

$$\therefore \frac{1 - \cos 2A}{a^2} = \frac{1 - \cos 2B}{b^2},$$

that is,

$$\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}.$$

$$191. \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{110 \cdot 33 \cdot 42 \cdot 35}$$

$$= \sqrt{11^2 \cdot 7^2 \cdot 3^2 \cdot 2^2 \cdot 5^2} = 11 \cdot 7 \cdot 3 \cdot 2 \cdot 5$$

$$= 2310 \text{ sq. ft.}$$

$$r_1 = \frac{\Delta}{s-a} = \frac{2310}{33} = 70 \text{ ft.}$$

$$r_2 = \frac{\Delta}{s-b} = \frac{2310}{42} = 55 \text{ ft.}$$

$$r_3 = \frac{\Delta}{s-c} = \frac{2310}{35} = 66 \text{ ft.}$$

192. Draw DK , DK' perpendicular to AB , AC respectively; then
 $DK \cdot AB + DK' \cdot AC = 2\Delta$;

that is, $c \cdot AD \sin \frac{A}{2} + b \cdot AD \sin \frac{A}{2} = bc \sin A$;

$$\therefore AD(b+c) = 2bc \cos \frac{A}{2}.$$

$$193. (1) \quad \sin 5\theta - \sin 3\theta = \sqrt{2} \sin \theta.$$

$$\therefore 2 \sin \theta \cos 4\theta = \sqrt{2} \sin \theta;$$

$$\text{that is,} \quad \sin \theta = 0, \text{ or } \cos 4\theta = \frac{1}{\sqrt{2}}.$$

$$\therefore \theta = n\pi, \text{ or } 4\theta = 2n\pi \pm \frac{\pi}{4}.$$

$$(2) \quad \cot \theta + \cot \left(\frac{\pi}{4} + \theta \right) = 2;$$

$$\therefore \cot \theta + \frac{\cot \theta - 1}{\cot \theta + 1} = 2;$$

$$\cot^2 \theta + 2 \cot \theta - 1 = 2 \cot \theta + 2;$$

$$\cot^2 \theta = 3;$$

$$\text{that is,} \quad \cot \theta = \pm \sqrt{3}, \text{ and } \theta = n\pi \pm \frac{\pi}{6}.$$

194. The given relation easily reduces to $\cos 2\alpha = \sin 2\beta$, one solution of which is $2\alpha = \frac{\pi}{2} - 2\beta$.

$$195. \text{ We have } \tan (\alpha + \theta) \tan (\alpha - \theta) = \tan^2 \beta;$$

$$\therefore \frac{\tan^2 \alpha - \tan^2 \theta}{1 - \tan^2 \alpha \tan^2 \theta} = \tan^2 \beta;$$

$$\text{whence } \tan^2 \theta (1 - \tan^2 \alpha \tan^2 \beta) = \tan^2 \alpha - \tan^2 \beta;$$

$$\begin{aligned} \therefore \tan^2 \theta &= \frac{(\tan \alpha + \tan \beta)(\tan \alpha - \tan \beta)}{(1 - \tan \alpha \tan \beta)(1 + \tan \alpha \tan \beta)} \\ &= \tan (\alpha + \beta) \tan (\alpha - \beta). \end{aligned}$$

$$196. (1) \text{ We have } p_1 = \frac{2\Delta}{a};$$

$$\therefore \text{ second side } = \frac{a^3 b^3 c^3}{8\Delta^3} = 8 \left(\frac{abc}{4\Delta} \right)^3 = 8R^3.$$

$$(2) \text{ Second side } = \frac{a^2 + b^2 - 2ab \cos C}{4\Delta^2}$$

$$= \frac{c^2}{4\Delta^2} = \left(\frac{c}{2\Delta} \right)^2 = \frac{1}{p_3^2}.$$

197. By Art. 215 we have

$$\text{perimeter} = 30r \tan \frac{\pi}{15}, \text{ where } \pi r^2 = 1386.$$

$$\text{Now } r^2 = \frac{7}{22} \times 1386 = 7 \times 63; \therefore r = 21;$$

$$\begin{aligned} \therefore \text{perimeter} &= 30 \times 21 \tan 12^\circ \\ &= 630 \times .213 \\ &= 134.19 \text{ ft.} \end{aligned}$$

198. Let A, B represent the foot of the pole in the two positions; C, S the top of the pole on the coping and sill respectively; also let W be the foot of the wall.

$$\begin{aligned} \text{Then } x + SW &= AC \sin \alpha, \\ \text{but } SW &= BS \sin \beta = AC \sin \beta; \\ \therefore x &= AC (\sin \alpha - \sin \beta). \end{aligned}$$

$$\text{Similarly } a = AC (\cos \beta - \cos \alpha);$$

$$\therefore \frac{x}{a} = \frac{2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}}{2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}} = \cot \frac{\alpha + \beta}{2}.$$

199. See Examples XVIII. *a.* 18. Each of the three expressions will be found to be equal to r .

$$200. (1) \text{ Let } \sin^{-1} \frac{2}{7} = \theta; \text{ then } \cos 2\theta = 1 - 2 \sin^2 \theta = \frac{41}{49};$$

$$\therefore \cos^{-1} \frac{41}{49} = 2\theta = 2 \sin^{-1} \frac{2}{7}.$$

$$\begin{aligned} (2) \quad 3 \tan^{-1} \frac{1}{4} &= \tan^{-1} \frac{3 \times \frac{1}{4} - \left(\frac{1}{4}\right)^3}{1 - 3 \left(\frac{1}{4}\right)^2} \\ &= \tan^{-1} \frac{3 \times 16 - 1}{64 - 12} = \tan^{-1} \frac{47}{52}. \end{aligned}$$

201. (1) As in XI. *f.* Ex. 14 we may prove that

$$\tan A + \sec A = \tan \left(45^\circ + \frac{A}{2} \right).$$

$$\text{Also } \cot A + \operatorname{cosec} A = \frac{1 + \cos A}{\sin A} = \cot \frac{A}{2};$$

$$\therefore (\tan A + \sec A) \cot \frac{A}{2} = (\cot A + \operatorname{cosec} A) \tan \left(45^\circ + \frac{A}{2} \right).$$

(2) First side

$$\begin{aligned}
 &= 2 \cos (A+B) \cos (A-B) - 2 \cos (A+B) \{ \cos (A-B) - \cos (90^\circ - \overline{A+B}) \} \\
 &= 2 \cos (A+B) \sin (A+B) = \sin 2 (A+B).
 \end{aligned}$$

202. By Art. 214, $\frac{\text{perimeter of fig.}}{\text{diameter of circle}} = 7 \sin 25\frac{5}{7}^\circ$,

$$\log 7 \sin 25\frac{5}{7}^\circ = .8450980 + \bar{1}.6373733$$

= .4824713, which is greater than log 3.

203. We have $c^2 = a^2 + b^2 - 2ab \cos C$

$$= (a^2 + b^2) \left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2} \right) - 2ab \left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2} \right)$$

$$= (a+b)^2 \sin^2 \frac{C}{2} + (a-b)^2 \cos^2 \frac{C}{2}$$

$$= (a+b)^2 \sin^2 \frac{C}{2} \left\{ 1 + \left(\frac{a-b}{a+b} \right)^2 \cot^2 \frac{C}{2} \right\}$$

$$= (a+b)^2 \sin^2 \frac{C}{2} (1 + \tan^2 \phi)$$

$$= (a+b)^2 \sin^2 \frac{C}{2} \sec^2 \phi;$$

$$\therefore c = (a+b) \sin \frac{C}{2} \sec \phi.$$

204. We have $\tan \phi = \frac{237-158}{237+158} \cot 33^\circ 20'$

$$= \frac{2}{10} \cot 33^\circ 20';$$

$$\therefore \log \tan \phi = \log 2 - 1 + \log \cot 33^\circ 20'$$

$$= \bar{1}.30103 + .18197$$

$$= \bar{1}.48300$$

$$\log \tan 16^\circ 54' = \bar{1}.48262$$

$$\text{diff.} \quad \quad \quad 38$$

$$\therefore \phi = 16^\circ 54' 50'',$$

$$\log \sec 16^\circ 55' = .01921$$

$$\log \sec 16^\circ 54' = .01917$$

$$\text{diff. for } 60'' \quad \quad \quad 4$$

$$\therefore \log \sec \phi = .01920.$$

$$\begin{aligned}
 \text{prop}^1 \text{ increase} &= \frac{38}{46} \times 60'' \\
 &= 50'';
 \end{aligned}$$

$$\begin{aligned}
 \text{prop}^1 \text{ increase for } 50'' &= \frac{4}{60} \times 50'' \\
 &= 3'';
 \end{aligned}$$

Now $c = (a + b) \sin \frac{C}{2} \sec \phi = 395 \sin \frac{C}{2} \sec \phi,$

$$\begin{aligned} \log 395 &= \log 79 + 1 - \log 2 \\ &= 2.59660 \end{aligned}$$

$$\log \sin \frac{C}{2} = 1.73998$$

$$\log \sec \phi = .01920$$

$$\log c = 2.35578$$

$$\therefore c = 226.87.$$

205.

$$2 \cos^2 2\theta = 1 + \cos 4\theta;$$

$$\therefore 2 \cos 2\theta = \sqrt{2 + 2 \cos 4\theta}.$$

Similarly

$$2 \cos \theta = \sqrt{2 + 2 \cos 2\theta} = \sqrt{2 + \sqrt{2 + 2 \cos 4\theta}}.$$

206. Let

$$\sin^{-1} \frac{3}{\sqrt{73}} = \alpha, \text{ so that } \cos \alpha = \frac{8}{\sqrt{73}},$$

and let

$$\cos^{-1} \frac{11}{\sqrt{146}} = \beta, \text{ so that } \sin \beta = \frac{5}{\sqrt{146}}.$$

Then

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$= \frac{3}{\sqrt{73}} \cdot \frac{11}{\sqrt{146}} + \frac{8}{\sqrt{73}} \cdot \frac{5}{\sqrt{146}} = \frac{73}{73\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$= \sin \frac{\pi}{4} = \sin \left(\frac{5\pi}{12} - \frac{\pi}{6} \right);$$

$$\therefore \alpha + \beta = \frac{5\pi}{12} - \frac{\pi}{6} = \frac{5\pi}{12} - \sin^{-1} \frac{1}{2}.$$

Again

$$\tan^{-1} \frac{\frac{x-1}{x+1} + \frac{2x-1}{2x+1}}{1 - \frac{x-1}{x+1} \cdot \frac{2x-1}{2x+1}} = \tan^{-1} \frac{23}{36};$$

$$\tan^{-1} \frac{4x^2 - 2}{6x} = \tan^{-1} \frac{23}{36};$$

$$\therefore 36(2x^2 - 1) = 69x,$$

or

$$24x^2 - 23x - 12 = 0;$$

$$\therefore (3x - 4)(8x + 3) = 0,$$

that is,

$$x = \frac{4}{3}, \text{ or } -\frac{3}{8}.$$

207. We have $x = \frac{2\Delta}{a}, R = \frac{abc}{4\Delta};$

$$\therefore x = \frac{1}{a} \cdot \frac{abc}{2R}; \quad \therefore \frac{bx}{c} = \frac{b^2}{2R},$$

that is,

$$\frac{bx}{c} + \frac{cy}{a} + \frac{az}{b} = \frac{a^2 + b^2 + c^2}{2R}.$$

208. We have $\cos(a + \theta) = \cos \left\{ \frac{\pi}{2} - (a - \theta) \right\},$

$$\therefore a + \theta = 2m\pi \pm \left\{ \frac{\pi}{2} - (a - \theta) \right\};$$

the upper sign gives $2a = 2m\pi + \frac{\pi}{2},$

and the lower sign gives $2\theta = 2m\pi - \frac{\pi}{2}.$

209. With the notation of Art. 228,

$$SI^2 = R^2 - 2Rr.$$

If a be the base of the triangle, $A = 120^\circ,$

$$B = C = 30^\circ; \quad \therefore r = 4R \sin 60^\circ \sin 15^\circ \sin 15^\circ;$$

$$\begin{aligned} \therefore SI^2 &= R^2 - 8R^2 \frac{\sqrt{3}}{2} \left(\frac{\sqrt{3} - 1}{2\sqrt{2}} \right)^2 \\ &= R^2 (4 - 2\sqrt{3}); \end{aligned}$$

$$\therefore SI = R(\sqrt{3} - 1)$$

$$= \frac{a}{2 \sin A} (\sqrt{3} - 1) = \frac{a(\sqrt{3} - 1)}{\sqrt{3}},$$

$$\therefore SI : a = \sqrt{3} - 1 : \sqrt{3}.$$

210. $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

$$= 1 + \frac{r}{R} = 1 + \frac{3}{4},$$

$$\therefore 4(\cos A + \cos B + \cos C) = 7.$$

211. Since $\frac{\cos(\theta - a)}{\sin(\theta + a)} = \frac{1 + m}{1 - m},$

dividendo and componendo, we have

$$\frac{\cos(\theta - a) - \sin(\theta + a)}{\cos(\theta - a) + \sin(\theta + a)} = m.$$

By expanding the sines and cosines we obtain

$$\frac{(\cos \theta - \sin \theta)(\cos \alpha - \sin \alpha)}{(\cos \theta + \sin \theta)(\cos \alpha + \sin \alpha)} = m,$$

or
$$\frac{1 - \tan \theta}{1 + \tan \theta} = m \left(\frac{\cot \alpha + 1}{\cot \alpha - 1} \right). \quad [\text{See XI. b. Ex. 6, 7.}]$$

212. (1)
$$\begin{aligned} \sin 5\theta - \sin 3\theta &= \sqrt{2} \cos 4\theta; \\ 2 \cos 4\theta \sin \theta &= \sqrt{2} \cos 4\theta; \\ \therefore \cos 4\theta &= 0; \text{ whence } 4\theta = (2n+1)\frac{\pi}{2}, \end{aligned}$$

or
$$\sin \theta = \frac{1}{\sqrt{2}}; \text{ whence } \theta = n\pi + (-1)^n \frac{\pi}{4}.$$

(2)
$$1 + \sin 2\theta = \frac{1 + \tan \theta}{1 - \tan \theta}.$$

$$1 + \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{1 + \tan \theta}{1 - \tan \theta}.$$

$$(1 + \tan \theta)^2 (1 - \tan \theta) = (1 + \tan \theta) (1 + \tan^2 \theta);$$

$$\therefore 1 + \tan \theta = 0; \text{ whence } \theta = n\pi + \frac{3\pi}{4},$$

or
$$1 - \tan^2 \theta = 1 + \tan^2 \theta;$$

$$\therefore \tan \theta = 0; \text{ whence } \theta = n\pi.$$

213. We have
$$2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} = 4 \sin^2 \frac{C}{2};$$

$$\therefore \cos \frac{A-B}{2} = 2 \sin \frac{C}{2}, \text{ or } 2 \cos \frac{A-B}{2} \sin \frac{A+B}{2} = 4 \sin \frac{C}{2} \cos \frac{C}{2};$$

that is,
$$\sin A + \sin B = 2 \sin C, \text{ or } a + b = 2c.$$

214. With the figure on p. 186, we have $\tan \beta = \frac{1}{9}$, $PA = 80$ ft., $CA = 100$ ft. Let $BP = x$ ft., then

$$\tan \theta = \frac{x+80}{100}, \quad \tan (\theta - \beta) = \frac{80}{100} = \frac{4}{5},$$

$$\therefore \frac{\tan \theta - \tan \beta}{1 + \tan \theta \tan \beta} = \frac{4}{5},$$

$$\frac{\frac{x+80}{100} - \frac{1}{9}}{1 + \frac{x+80}{900}} = \frac{4}{5};$$

$$\therefore 5(9x + 720 - 100) = 4(980 + x);$$

$$45x + 3100 = 3920 + 4x;$$

$$41x = 820; \text{ or } x = 20.$$

$$215. (1) \quad \cot^{-1} 7 + \cot^{-1} 8 = \cot^{-1} \frac{7 \cdot 8 - 1}{7 + 8} = \cot^{-1} \frac{55}{15},$$

$$\cot^{-1} 3 - \cot^{-1} 18 = \cot^{-1} \frac{3 \cdot 18 + 1}{18 - 3} = \cot^{-1} \frac{55}{15}.$$

$$(2) \quad 4 \tan^{-1} \frac{1}{5} = 2 \tan^{-1} \frac{\frac{2}{5}}{1 - \frac{1}{5^2}} = 2 \tan^{-1} \frac{5}{12}$$

$$= \tan^{-1} \frac{\frac{2 \times 5}{12}}{1 - \frac{25}{12^2}} = \tan^{-1} \frac{120}{119}.$$

$$\begin{aligned} \therefore 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} &= \tan^{-1} \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}} = \tan^{-1} \frac{120 \cdot 239 - 119}{119 \cdot 239 + 120} \\ &= \tan^{-1} \frac{119 \cdot 239 + (239 - 119)}{119 \cdot 239 + 120} = \tan^{-1} 1 = \frac{\pi}{4}. \end{aligned}$$

216. Proceeding as in Art. 259, Ex. 2, we find that $\frac{A}{2}$ lies between $2n\pi + \frac{3\pi}{4}$ and $2n\pi + \frac{5\pi}{4}$; that is A lies between $(8n+3)\frac{\pi}{2}$ and $(8n+5)\frac{\pi}{2}$.

$$\begin{aligned} 217. \text{ First side } &= \frac{1}{2} \left\{ 1 - \cos \left(\frac{\pi}{4} + \theta \right) - 1 + \cos \left(\frac{\pi}{4} - \theta \right) \right\} \\ &= \sin \frac{\pi}{4} \sin \theta = \frac{1}{\sqrt{2}} \sin \theta. \end{aligned}$$

218. From the two given relations we easily deduce

$$x = \frac{\sin \theta}{\sin (\theta + \phi)}, \quad y = \frac{\sin \phi}{\sin (\theta + \phi)},$$

$$\therefore \sin \theta : \sin \phi = x : y.$$

$$219. \quad \tan^{-1} \frac{x+1+x-1}{1-(x^2-1)} = \tan^{-1} \frac{8}{31};$$

$$\therefore \frac{2x}{2-x^2} = \frac{8}{31}, \text{ or } 4x^2 + 31x - 8 = 0;$$

$$\therefore (4x-1)(x+8) = 0, \text{ or } x = \frac{1}{4}, \text{ or } -8.$$

Again $\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$
 $= 1 + \tan^2(\tan^{-1} 2) + 1 + \cot^2(\cot^{-1} 3)$
 $= 1 + 4 + 1 + 9 = 15.$

220. Let $5A = \alpha$, $5B = \beta$, $5C = \gamma$, then $\alpha + \beta + \gamma = 5\pi$;

$$\begin{aligned}\therefore \sin(\alpha + \beta) &= \sin(\pi - \gamma) = \sin \gamma; \cos \gamma = -\cos(\alpha + \beta), \\ \sin 2\alpha + \sin 2\beta + \sin 2\gamma &= 2 \sin(\alpha + \beta) \cos(\alpha - \beta) + 2 \sin \gamma \cos \gamma \\ &= 2 \sin \gamma \{\cos(\alpha - \beta) - \cos(\alpha + \beta)\} \\ &= 4 \sin \alpha \sin \beta \sin \gamma.\end{aligned}$$

In the second case the sum of the three angles is $\frac{16\pi}{2^5}$, or $\frac{\pi}{2}$ and the result easily follows as in Art. 135, Ex. 2.

221. See solution to XVIII. a. Ex. 24.

222. We have $x = \frac{BD \sin 15^\circ}{\sin 50^\circ}$; $BD = \frac{100}{\cos 25^\circ}$;

$$\therefore x = \frac{100 \cos 75^\circ}{\cos 40^\circ \cos 25^\circ};$$

$$\begin{aligned}\therefore \log x &= 2 + \log \cos 75^\circ - (\log \cos 40^\circ + \log \cos 25^\circ) \\ &= 1.4129962 - (1.8415297)\end{aligned}$$

$$= 1.5714665$$

$$\log 37.279 = 1.5714643$$

$$\text{diff.} \quad \underline{\quad\quad\quad} \quad 22$$

$$\begin{aligned}\text{prop}^l. \text{ increase} &= \frac{22}{116} \times .001 \\ &= .00019;\end{aligned}$$

$$\therefore x = 37.27919.$$

$$\begin{aligned}223. \quad \{\sec \theta + \operatorname{cosec} \theta (1 + \sec \theta)\}^2 &= \left(\frac{1}{\cos \theta} + \frac{1 + \cos \theta}{\sin \theta \cos \theta} \right)^2 \\ &= \frac{(1 + \sin \theta + \cos \theta)^2}{\sin^2 \theta \cos^2 \theta} \\ &= \frac{2 + 2 \sin \theta + 2 \cos \theta + 2 \sin \theta \cos \theta}{\sin^2 \theta \cos^2 \theta};\end{aligned}$$

$$\begin{aligned}\therefore \text{First side} &= 2 \sec^2 \theta \frac{(1 - \cos \theta)}{\sin^2 \theta} (1 + \sin \theta + \cos \theta + \sin \theta \cos \theta) \\ &= \frac{2 \sec^2 \theta (1 + \cos \theta) (1 + \sin \theta)}{1 + \cos \theta} \\ &= 2 \sec^2 \theta (1 + \sin \theta).\end{aligned}$$

224. The relation given will be true if

$$\frac{1}{a+b+c} - \frac{1}{a+c} = \frac{1}{b+c} - \frac{2}{a+b+c};$$

i.e. if

$$-\frac{b}{a+c} = \frac{a-b-c}{b+c}, \text{ or } \frac{b}{a+c} = \frac{b+c-a}{b+c},$$

i.e. if

$$b(b+c) = (b+c)(a+c) - a^2 - ac.$$

From this we easily deduce $\frac{a^2+b^2-c^2}{ab} = 1$, which is true when $C = 60^\circ$.

225. The solution of this example is merely an extension of that of Ex. 205.

226. We have $m \sin(a-\theta) \cos(a-\theta) = n \sin \theta \cos \theta$,

$$m \sin 2(a-\theta) = n \sin 2\theta;$$

$$\therefore \frac{\sin 2(a-\theta) - \sin 2\theta}{\sin 2(a-\theta) + \sin 2\theta} = \frac{n-m}{n+m},$$

$$\frac{\sin(a-2\theta) \cos a}{\cos(a-2\theta) \sin a} = \frac{n-m}{n+m},$$

$$\tan(a-2\theta) = \frac{n-m}{n+m} \tan a,$$

$$a-2\theta = \tan^{-1} \left(\frac{n-m}{n+m} \tan a \right),$$

$$\theta = \frac{1}{2} \left\{ a - \tan^{-1} \left(\frac{n-m}{n+m} \tan a \right) \right\}.$$

227. Put k for each of the equal ratios, then it easily follows that $s = k(1+n^2)$.

$$\text{Now } \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{(1-m^2)m^2(1+n^2)}{(1+n^2)n^2(1-m^2)}} = \frac{m}{n};$$

$$\therefore A = 2 \tan^{-1} \frac{m}{n}; \text{ similarly } B = 2 \tan^{-1} mn.$$

Again,

$$\begin{aligned} \Delta &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= k^2 \sqrt{(1+n^2)^2 (1-m^2)^2 m^2 n^2} \\ &= k^2 (1-m^2)(1+n^2) mn = kcmn \\ &= \frac{mbc}{m^2+n^2}, \text{ since } \frac{b}{m^2+n^2} = k. \end{aligned}$$

228. See figure and solution of Example II. page 190.

$$\begin{aligned}\text{Here} \quad h &= CD = \frac{a \sin \beta}{\cos (2\alpha + \beta)}, \\ l &= DE = \frac{a \sin \alpha \cos (\alpha + \beta)}{\cos (2\alpha + \beta)}.\end{aligned}$$

$$\begin{aligned}\text{But} \quad 2\alpha + \beta + \theta &= 90^\circ; \therefore \cos (2\alpha + \beta) = \sin \theta; \\ \therefore \text{by substitution,} \quad h &= a \sin \beta \operatorname{cosec} \theta, \\ 2l &= 2a \operatorname{cosec} \theta \sin \alpha \cos (\alpha + \beta) \\ &= a \operatorname{cosec} \theta \{ \sin (2\alpha + \beta) - \sin \beta \} \\ &= a \operatorname{cosec} \theta (\cos \theta - \sin \beta).\end{aligned}$$

229. We have

$$\begin{aligned}\frac{1}{2} \tan \frac{\theta}{2} + \cot \theta &= \frac{1}{2} \cdot \frac{1 - \cos \theta}{\sin \theta} + \frac{\cos \theta}{\sin \theta} = \frac{1 - \cos \theta + 2 \cos \theta}{2 \sin \theta} = \frac{1 + \cos \theta}{2 \sin \theta} \\ &= \frac{2 \cos^2 \frac{\theta}{2}}{4 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{\cos \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} \\ &= \frac{\cos^2 \frac{\theta}{4} - \sin^2 \frac{\theta}{4}}{4 \sin \frac{\theta}{4} \cos \frac{\theta}{4}} = \frac{1}{4} \cot \frac{\theta}{4} - \frac{1}{4} \tan \frac{\theta}{4}.\end{aligned}$$

230. With the figure of Art. 214 we have $AD = \frac{AO}{2m}$.

Let $\theta = \angle AOD$, then $\cos AOB = 1 - 2 \sin^2 \theta$

$$\begin{aligned}&= 1 - \frac{2AD^2}{AO^2} = 1 - \frac{1}{2m^2} = \frac{2m^2 - 1}{2m^2}; \\ \therefore AOB &= \sec^{-1} \frac{2m^2}{2m^2 - 1}.\end{aligned}$$

231. With the figure of Art 268, Ex. 1, we have

$$\begin{aligned}\frac{PN}{ON} &= \tan 1'' = \text{radian measure of } 1'', \text{ approx.} \\ &= \frac{\pi}{180} \times \frac{1}{60} \times \frac{1}{60};\end{aligned}$$

$$\begin{aligned}\therefore ON &= \frac{180 \times 60 \times 60}{\pi} \text{ inches} \\ &= \frac{180 \times 60 \times 60}{1760 \times 3 \times 12 \pi} \text{ miles} \\ &= \frac{1800}{176\pi} \text{ miles} = 3\frac{1}{4} \text{ miles, nearly.}\end{aligned}$$

$$\begin{aligned}232. \quad \tan^{-1} y &= 2 \tan^{-1} \frac{2x}{1-x^2} = \tan^{-1} \frac{\frac{4x}{1-x^2}}{1 - \frac{4x^2}{(1-x^2)^2}} \\ &= \tan^{-1} \frac{4x(1-x^2)}{1-6x^2+x^4}; \\ \therefore y &= \frac{4x(1-x^2)}{1-6x^2+x^4}.\end{aligned}$$

If $y = \tan \frac{\pi}{2}, \quad 1-6x^2+x^4=0,$

but $x = \tan \frac{1}{4} (\tan^{-1} y) = \tan \frac{\pi}{8},$

thus $\tan \frac{\pi}{8}$ is a root of $x^4 - 6x^2 + 1 = 0.$

233. We have $\tan^2 a = \frac{1 - \cos 2a}{1 + \cos 2a} = \frac{49}{529};$

$$\therefore \tan a = \pm \frac{7}{23}.$$

The two values may be explained as in Art. 261, Ex. 2.

234. We have $\frac{\sin \theta}{a} = \frac{\sin \phi}{b} = \frac{\sin (\theta + \phi)}{c};$

$$\therefore \frac{\sin \theta + \sin \phi}{a + b} = \frac{\sin (\theta + \phi)}{c}.$$

But $a + b = 2c,$

$$\therefore \sin \theta + \sin \phi = 2 \sin (\theta + \phi),$$

whence $\cos \frac{\theta - \phi}{2} = 2 \cos \frac{\theta + \phi}{2} \dots\dots\dots(1),$

or $\cos \frac{\theta + \phi}{2} = \cos \frac{\theta - \phi}{2} - \cos \frac{\theta + \phi}{2}$
 $= 2 \sin \frac{\theta}{2} \sin \frac{\phi}{2} \dots\dots\dots(2).$

Now
$$\begin{aligned}\cos \theta + \cos \phi &= 2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} \\ &= 2 \left(2 \sin \frac{\theta}{2} \sin \frac{\phi}{2} \right) 2 \cos \frac{\theta + \phi}{2}, \text{ by (1) and (2)} \\ &= 4 \sin \frac{\theta}{2} \sin \frac{\phi}{2} \cdot 4 \sin \frac{\theta}{2} \sin \frac{\phi}{2}, \text{ by (2)} \\ &= 16 \sin^2 \frac{\theta}{2} \sin^2 \frac{\phi}{2} \\ &= 4 (1 - \cos \theta) (1 - \cos \phi).\end{aligned}$$

235. (1)
$$\begin{aligned}\sin 7\theta + \sin \theta &= \sin 4\theta; \\ \therefore 2 \sin 4\theta \cos 3\theta &= \sin 4\theta; \\ \therefore \text{either } \sin 4\theta &= 0, \text{ or } \cos 3\theta = \frac{1}{2};\end{aligned}$$

that is,
$$4\theta = n\pi, \text{ or } 3\theta = 2n\pi \pm \frac{\pi}{3}.$$

(2)
$$\begin{aligned}\tan x - \frac{\sqrt{3}}{\tan x} + 1 - \sqrt{3} &= 0; \\ \tan^2 x - (\sqrt{3} - 1) \tan x - \sqrt{3} &= 0; \\ (\tan x - \sqrt{3})(\tan x + 1) &= 0; \\ \therefore \text{either } \tan x &= \sqrt{3}, \text{ or } \tan x = -1;\end{aligned}$$

that is,
$$x = n\pi + \frac{\pi}{3}, \text{ or } x = n\pi + \frac{3\pi}{4}.$$

236. (1)
$$\begin{aligned}\sin 3A &= \sin 3(180^\circ - \overline{B + C}) \\ &= \sin (360^\circ + 180^\circ - 3\overline{B + C}) = \sin 3\overline{B + C}.\end{aligned}$$

We have only now to prove that

$$\Sigma \sin 3(B + C) \sin(B - C) = 0,$$

and this follows by separating each term into the difference of two cosines.

(2) It will be sufficient to prove that

$$\Sigma \sin^3 A \sin(B - C) = 0.$$

Now
$$\sin^3 A = \frac{3}{4} \sin A - \frac{1}{4} \sin 3A;$$

$$\begin{aligned}\therefore \Sigma \sin^3 A \sin(B - C) &= \frac{3}{4} \Sigma \sin A \sin \overline{B - C} - \frac{1}{4} \Sigma \sin 3A \sin \overline{B - C} \\ &= 0,\end{aligned}$$

by the first part of the question.

237. The angle $ACP = \theta - A$.

$$\therefore \frac{AP}{PC} = \frac{\sin(\theta - A)}{\sin A} = \sin \theta \cot A - \cos \theta,$$

$$\frac{PC}{PB} = \frac{\sin B}{\sin(\theta + B)} = \frac{1}{\sin \theta \cot B + \cos \theta};$$

$\therefore$ by multiplication

$$\frac{AP}{PB} = \frac{m}{n} = \frac{\sin \theta \cot A - \cos \theta}{\sin \theta \cot B + \cos \theta};$$

whence $\sin \theta (n \cot A - m \cot B) = (m + n) \cos \theta$,

or $(m + n) \cot \theta = n \cot A - m \cot B$.

238. The equation may be written

$$a \sin \theta - \cos \theta + b = 0 \dots\dots\dots (1).$$

Since α and β are roots of this equation

$$a \sin \alpha - \cos \alpha + b = 0,$$

$$a \sin \beta - \cos \beta + b = 0,$$

whence a and b may be found.

Again from (1),

$$(a \sin \theta + b)^2 = 1 - \sin^2 \theta.$$

or $(1 + a^2) \sin^2 \theta + 2ab \sin \theta + b^2 - 1 = 0;$

since α, β are roots of this equation,

$$\sin \alpha + \sin \beta = -\frac{2ab}{1 + a^2}.$$

Similarly we may shew that $\cos \alpha + \cos \beta = \frac{2b}{1 + a^2}$, whence the required result follows.

239. Write s and c for $\sin \theta$ and $\cos \theta$ respectively; then

$$\frac{u_3 - u_5}{u_1} = \frac{s^3 + c^3 - (s^5 + c^5)}{s + c} = \frac{s^3(1 - s^2) + c^3(1 - c^2)}{s + c} = \frac{s^3c^2 + c^3s^2}{s + c} = s^2c^2.$$

Again
$$\frac{u_5 - u_7}{u_3} = \frac{s^5 + c^5 - (s^7 + c^7)}{s^3 + c^3} = \frac{s^5(1 - s^2) + c^5(1 - c^2)}{s^3 + c^3}$$

$$= \frac{s^5c^2 + c^5s^2}{s^3 + c^3} = s^2c^2.$$

240. Let E, F be the first and second points of observation respectively; then $EF=a$, and EAD is a straight line. Let x =a side of the square base, then $EA=AB=AD=x$. Then if $\angle AFD=\theta$, we have

$$AD^2=AF^2+FD^2-2AF \cdot FD \cos \theta \dots\dots\dots (1).$$

But $AD^2=x^2$, $AF^2=x^2+a^2$, $FD^2=4x^2+a^2$. Also $\cos \theta=\frac{1}{3}\sqrt{8}$. Substituting these values in (1) we obtain $x=\frac{a\sqrt{2}}{2}$.

$$\begin{aligned} 241. \quad (1) \quad \text{First side} &= \frac{1-\sin^2 A}{\sin A} \cdot \frac{1-\cos^2 A}{\cos A} = \frac{\cos^2 A \sin^2 A}{\sin A \cos A} \\ &= \sin A \cos A. \end{aligned}$$

$$\text{Again, second side} = \left(\frac{\sin^2 A + \cos^2 A}{\sin A \cos A} \right)^{-1} = \sin A \cos A.$$

$$\begin{aligned} (2) \quad \text{First side} &= \frac{\tan \theta}{\sec^4 \theta} + \frac{\cot \theta}{\operatorname{cosec}^4 \theta} \\ &= \sin \theta \cos^3 \theta + \cos \theta \sin^3 \theta \\ &= \sin \theta \cos \theta (\sin^2 \theta + \cos^2 \theta) \\ &= \frac{1}{2} \sin 2\theta. \end{aligned}$$

$$242. \quad \text{We have} \quad 2 \sin 4\theta \cos \theta = \frac{1}{2} + 2 \sin \frac{5\theta}{2} \cos \frac{5\theta}{2};$$

$$\therefore \sin 5\theta + \sin 3\theta = \frac{1}{2} + 2 \sin \frac{5\theta}{2} \cos \frac{5\theta}{2};$$

$$\therefore \sin 3\theta = \frac{1}{2},$$

one solution of which is $3\theta=30^\circ$.

$$243. \quad \tan^{-1} \frac{2mn}{m^2-n^2} = \tan^{-1} \frac{2 \frac{n}{m}}{1-\frac{n^2}{m^2}} = 2 \tan^{-1} \frac{n}{m};$$

$$\begin{aligned} \therefore \text{first side} &= 2 \tan^{-1} \frac{n}{m} + 2 \tan^{-1} \frac{q}{p} \\ &= 2 \left(\tan^{-1} \frac{n}{m} + \tan^{-1} \frac{q}{p} \right) \\ &= 2 \tan^{-1} \frac{\frac{n}{m} + \frac{q}{p}}{1 - \frac{nq}{mp}}; \end{aligned}$$

$$\begin{aligned}\therefore \text{first side} &= 2 \tan^{-1} \frac{np+mq}{mp-nq} = 2 \tan^{-1} \frac{N}{M} \\ &= \tan^{-1} \frac{2MN}{M^2 - N^2}.\end{aligned}$$

244. Write $\frac{r}{s-a}$ for $\tan \frac{A}{2}$, then the first side becomes

$$\frac{r}{(s-a)(a-b)(a-c)} + \text{two similar terms.}$$

$$\begin{aligned}\text{Now} \quad \frac{r}{(s-a)(a-b)(a-c)} &= -\frac{rs(b-c)(s-b)(s-c)}{\Delta^2(a-b)(b-c)(c-a)} \\ &= -\frac{1}{\Delta} \frac{(b-c)\{s^2-s(b+c)+bc\}}{(a-b)(b-c)(c-a)}.\end{aligned}$$

Now $\Sigma(b-c)\{s^2-s(b+c)+bc\} = -(a-b)(b-c)(c-a)$.
[See Hall and Knight's Elem. Algebra, Art. 224.]

Thus the first side reduces to $\frac{1}{\Delta}$.

$$245. \quad \text{We have} \quad \left(1 - \frac{s-b}{s-a}\right) \left(1 - \frac{s-c}{s-a}\right) = 2;$$

$$\begin{aligned}\text{or} \quad (b-a)(c-a) &= 2(s-a)^2; \\ bc - ac - ab + a^2 &= 2s^2 - 4as + 2a^2; \\ \therefore bc &= 2s^2 - 4as + a(a+b+c) \\ &= 2s^2 - 4as + 2as;\end{aligned}$$

$$\text{whence} \quad \frac{s(s-a)}{bc} = \frac{1}{2};$$

$$\text{that is,} \quad \cos \frac{A}{2} = \frac{1}{\sqrt{2}}, \text{ or } A = 90^\circ.$$

$$246. \quad \text{Let} \quad A = 58^\circ 40' 3.9'', \quad b = 237, \quad c = 158.$$

Then as in Art. 197 we obtain

$$\begin{aligned}a &= (b+c) \sin \theta, \text{ where } \cos \theta = \frac{2\sqrt{bc}}{b+c} \cos \frac{A}{2}, \\ \cos \theta &= \frac{2\sqrt{237 \times 158}}{395} \cos 29^\circ 20' 1.95'' \\ &= \frac{2\sqrt{6}}{5} \cos 29^\circ 20' 1.95'',\end{aligned}$$

$$\begin{array}{rcl}
 \log 2 & = & \cdot 3010300 \\
 \log 3 & = & \cdot 4771213 \\
 & 2 & \overline{) \cdot 7781513} \\
 \frac{1}{2} \log 6 & = & \cdot 3890757 \\
 2 \log 2 - 1 & = & \bar{1} \cdot 6020600 \\
 \log \cos 29^\circ 20' & = & \bar{1} \cdot 9404091 \\
 \text{Subtract diff. for } 1 \cdot 95'' & & \underline{23} \\
 \log \cos \theta & = & \bar{1} \cdot 9315425 \\
 \log \cos 31^\circ 20' & = & \bar{1} \cdot 9315374 \\
 & & \underline{51} \\
 & & \frac{51}{769} \times 60'' = 3 \cdot 97'' \\
 \therefore \theta & = & 31^\circ 19' 56''.
 \end{array}$$

Again

$$\begin{array}{rcl}
 \log 395 & = & 2 \cdot 5965971 \\
 \log \sin 31^\circ 19' & = & \bar{1} \cdot 7158092 \\
 \text{diff. for } 56'' & & \underline{1937} \\
 \log a & = & \bar{2} \cdot 3126000 \\
 \log 205 \cdot 4 & = & \bar{2} \cdot 3126004 \\
 \therefore a & = & 205 \cdot 4.
 \end{array}$$

247. We have

$$\frac{\sin(A+B)}{\cos(A+B)} = \frac{3 \sin A}{\cos A};$$

whence

$$\sin(A+B) \cos A = 3 \cos(A+B) \sin A,$$

or

$$\sin(2A+B) + \sin B = 3 \sin(2A+B) - 3 \sin B;$$

that is

$$2 \sin(2A+B) = 4 \sin B.$$

Multiply by $\cos B$; then by separating the product on the left into the sum of two sines we obtain the required result.

$$\begin{aligned}
 248. \quad \text{First side} &= 2 \sin(\theta - a) \{ \sin(\theta - a) + \sin(2m\theta - a - \theta) \} \\
 &= 2 \sin^2(\theta - a) + \cos(2\theta - 2m\theta) - \cos(2m\theta - 2a) \\
 &= 1 - \cos(2\theta - 2a) + \cos(2\theta - 2m\theta) - \cos(2m\theta - 2a).
 \end{aligned}$$

249. Let AD be perpendicular to BC and meet the circum-circle in E ; then $\angle BED = C$, and $a = DE$.

$$\text{Now} \quad \frac{BD}{a} = \tan C, \text{ and } \frac{DC}{a} = \tan B;$$

$$\therefore \frac{BD + DC}{a} = \frac{a}{a} = \tan B + \tan C.$$

$$\text{Similarly} \quad \frac{b}{\beta} = \tan C + \tan A, \quad \frac{c}{\gamma} = \tan A + \tan B;$$

whence the result follows.

250. From the first equation, $3 \sin^2 A = 1 - 2 \sin^2 B = \cos 2B$; and from the second equation,

$$6 \sin A \cos A - 2 \sin 2B = 0;$$

multiply each term by $\sin A$, and we have

$$3 \sin^2 A \cos A - \sin 2B \sin A = 0.$$

Substituting $\cos 2B$ for $3 \sin^2 A$, we obtain

$$\cos 2B \cos A - \sin 2B \sin A = 0;$$

that is,

$$\cos (A + 2B) = 0; \text{ or } A + 2B = 90^\circ.$$

$$251. (1) \text{ First side} = \cot^{-1} \left(\frac{1}{\cot 2x} \right) + \cot^{-1} \left(-\frac{1}{\cot 3x} \right)$$

$$= \cot^{-1} \frac{\frac{1}{\cot 2x} \left(-\frac{1}{\cot 3x} \right) - 1}{-\frac{1}{\cot 3x} + \frac{1}{\cot 2x}}$$

$$= \cot^{-1} \left(\frac{\cot 3x \cot 2x + 1}{\cot 2x - \cot 3x} \right)$$

$$= \cot^{-1} (\cot x) = x.$$

$$(2) \text{ First side} = \tan^{-1} \frac{\frac{1-x}{1+x} - \frac{1-y}{1+y}}{1 + \frac{(1-x)(1-y)}{(1+x)(1+y)}} = \tan^{-1} \frac{2(y-x)}{2(1+xy)}$$

$$= \sin^{-1} \frac{\frac{y-x}{1+xy}}{\sqrt{1 + \frac{(y-x)^2}{(1+xy)^2}}} = \sin^{-1} \frac{y-x}{\sqrt{1+x^2} \sqrt{1+y^2}}.$$

252. See Art. 197.

Let A be the position of the station, B and C the positions of the two points; then $A = 49^\circ 45'$, $c = 1250$ yds., $b = 1575$ yds.

$$\text{Now } a = 2825 \cos \theta, \text{ where } \sin \theta = \frac{2 \sqrt{1250 \times 1575}}{2825} \cos 24^\circ 52' 30'',$$

$$\log 1250 = 3.0969100$$

$$\log 1575 = 3.1972806$$

$$2 \mid 6.2941906$$

$$3.1470953$$

$$\log 2 = .3010300$$

$$\log \cos 24^\circ 52' 30'' = \bar{1}.9577163$$

$$3.4058416$$

$$\log 2825 = 3.4510185$$

$$\log \sin \theta = \bar{1}.9548231 = \log \sin 64^\circ 19'.$$

$$\log \cos 24^\circ 52' = \bar{1}.9577456$$

$$\text{subtract } \frac{1}{2} \times 586 \quad \underline{293}$$

$$\bar{1}.9577163$$

Again	$\log 2825$	$= 3.4510185$
	$\log \cos 64^\circ 19'$	$= \bar{1}.6368859$
	$\log a$	$= 3.0879044$
	$\log 1224.3$	$= 3.0878878$
		166
	4	142
		240
	7	249

$\therefore a = 1224.347$ yards.

254. Multiply all through by 2; then

First side $= 1 + \cos 2S + 1 + \cos 2(S - A) +$ two similar terms

$$= 4 + 2 \cos (2S - A) \cos A + 2 \cos (2S - B - C) \cos (B - C)$$

$$= 4 + 2 \cos (B + C) \cos A + 2 \cos A \cos (B - C)$$

$$= 4 + 4 \cos A \cos B \cos C.$$

255. It is easy to see that this is the same as Example 1 in Art. 135.

256. We have $R = \frac{a}{2 \sin A} = 18 \operatorname{cosec} 61^\circ 15'.$

$\log 18$	$= 1.2552725$
$\log \operatorname{cosec} 61^\circ 15'$	$= .0571357$
$\log R$	$= \bar{1}.3124082$
$\log 20.530$	$= 1.3123889$
	193
9	191

$\therefore R = 20.5309.$

Again, $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$

$\log R$	$= 1.3124082$
$\log 4$	$= .6020600$
$\log \sin 30^\circ 37'$	$= \bar{1}.7069667$
diff. for $30''$	$= 1067$
$\log \sin 36^\circ 37'$	$= \bar{1}.7755801$
diff. for $30''$	$= 850$
$\log \sin 22^\circ 45'$	$= \bar{1}.5873865$
$\therefore \log r$	$= .9845932$
$\log 9.6514$	$= .9845903$
	29
6	27

$\therefore r = 9.65146.$

257. This follows from XVIII. *c.* Ex. 5 and XII. *d.* Ex. 12.

258. Let $\angle APB = \alpha$, $\angle BPC = \beta$, $\angle PBC = \gamma$;
 then $\frac{PB}{AB} = \frac{\sin(\gamma - \alpha)}{\sin \alpha}$, $\frac{PB}{BC} = \frac{\sin(\beta + \gamma)}{\sin \beta}$;
 but $AB = BC$,
 $\therefore \frac{\sin(\gamma - \alpha)}{\sin \alpha} = \frac{\sin(\beta + \gamma)}{\sin \beta}$,
 or $\sin \gamma \cot \alpha - \cos \gamma = \cos \gamma + \sin \gamma \cot \beta$;
 $\therefore 2 \cos \gamma = \sin \gamma (\cot \alpha - \cot \beta)$,
 $2 \cot \gamma = \cot \alpha - \cot \beta$;
 that is,
 $\frac{2}{T} = \frac{1}{t'} - \frac{1}{t}$,
 since γ is the supplement of the angle BP makes with the road.

$$\begin{aligned}
 259. \text{ First side} &= \frac{(\cos B + \cos C)(1 + 2 \cos A)}{(1 + 2 \cos A)(1 - \cos A)} = \frac{2 \cos \frac{B+C}{2} \cos \frac{B-C}{2}}{2 \sin^2 \frac{A}{2}} \\
 &= \frac{2 \cos \frac{B-C}{2}}{2 \sin \frac{A}{2}} \cdot \frac{\cos \frac{A}{2}}{\cos \frac{A}{2}} = \frac{2 \cos \frac{B-C}{2} \sin \frac{B+C}{2}}{\sin A} \\
 &= \frac{\sin B + \sin C}{\sin A} = \frac{b+c}{a}.
 \end{aligned}$$

$$\begin{aligned}
 260. \text{ From the fig. of Art. 219, we have } \frac{AI}{AI_1} &= \frac{s-a}{s}; \\
 \therefore \text{ first side} &= \frac{s-a}{s} + \frac{s-b}{s} + \frac{s-c}{s} = \frac{3s-2s}{s} = 1
 \end{aligned}$$

$$\begin{aligned}
 261. \text{ Let } \sin^{-1} \frac{1}{3} &= \alpha, \text{ then } \cos \alpha = \frac{2\sqrt{2}}{3}; \\
 \sin^{-1} \frac{3}{\sqrt{11}} &= \beta, \text{ then } \cos \beta = \frac{\sqrt{2}}{\sqrt{11}}.
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \sin(\alpha + \beta) &= \frac{1}{3} \cdot \frac{\sqrt{2}}{\sqrt{11}} + \frac{2\sqrt{2}}{3} \cdot \frac{3}{\sqrt{11}} = \frac{7\sqrt{2}}{3\sqrt{11}} \\
 &= \cos \left(\sin^{-1} \frac{1}{3\sqrt{11}} \right), \\
 \therefore \alpha + \beta &= \frac{\pi}{2} - \sin^{-1} \frac{1}{3\sqrt{11}},
 \end{aligned}$$

$$\text{or } \sin^{-1} \frac{1}{3} + \sin^{-1} \frac{1}{3\sqrt{11}} + \sin^{-1} \frac{3}{\sqrt{11}} = \frac{\pi}{2}.$$

262. We have $\cot \alpha (\cot \beta \cot \gamma - 1) = \cot \beta + \cot \gamma$,

$$\therefore \cot (\beta + \gamma) = \frac{1}{\cot \alpha} = \frac{1}{\tan \alpha}$$

$$= \cot \left(\frac{\pi}{2} - \alpha \right);$$

$$\therefore \beta + \gamma = n\pi + \frac{\pi}{2} - \alpha,$$

or

$$\alpha + \beta + \gamma = (2n+1) \frac{\pi}{2}.$$

263. Since $\tan A + \tan B + \tan C = \tan A \tan B \tan C$,

$$\text{the first side} = \frac{\tan^2 A + \tan^2 B + \tan^2 C}{\tan A + \tan B + \tan C}$$

$$= \frac{(\tan A + \tan B + \tan C)^2 - 2 \tan A \tan B - 2 \tan B \tan C - 2 \tan C \tan A}{\tan A + \tan B + \tan C}$$

$$= \tan A + \tan B + \tan C - \frac{2 (\tan A \tan B + \dots + \dots)}{\tan A \tan B \tan C}$$

$$= \tan A + \tan B + \tan C - 2 (\cot A + \cot B + \cot C).$$

264. Let O_1, O_2 be the two points of observation, A and B the two objects, so that $\angle AO_1O_2 = 45^\circ$, $\angle O_2O_1 = O_1O_2B = 22\frac{1}{2}^\circ$. Then $\angle O_1AO_2 = 112\frac{1}{2}^\circ$, $\angle O_1BO_2 = 22\frac{1}{2}^\circ$, and $O_1B = O_1O_2 = 1$ mile.

$$\text{Now from the } \triangle O_1AO_2, \frac{O_1A}{1 \text{ mile}} = \frac{\sin 22\frac{1}{2}^\circ}{\sin 112\frac{1}{2}^\circ} = \tan 22\frac{1}{2}^\circ = \sqrt{2} - 1;$$

$$\therefore O_1A = \sqrt{2} - 1 \text{ miles}; \therefore AB = \sqrt{2} \text{ miles.}$$

Again, if p_1, p_2 be the perpendiculars from AB on O_1O_2

$$p_1 + p_2 = (O_1A + O_1B) \sin 45^\circ = AB \sin 45^\circ = 1 \text{ mile.}$$

265. This follows from the identity

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C,$$

where $A + B + C = 180^\circ$, by putting $A = 20^\circ$, $B = 40^\circ$, $C = 120^\circ$.

266. The equation may be written

$$(2 \operatorname{cosec} 2\theta)^3 = 3 (2 \operatorname{cosec} 2\theta) + \frac{\cos^3 \theta}{\sin^3 \theta},$$

or

$$\left(\frac{1}{\sin \theta \cos \theta} \right)^3 = \frac{3}{\sin \theta \cos \theta} + \frac{\cos^3 \theta}{\sin^3 \theta};$$

that is,

$$1 = 3 \sin^2 \theta \cos^2 \theta + \cos^6 \theta,$$

which reduces to $(\cos^2 \theta - 1)^3 = 0$, whence $\theta = n\pi$.

267. When $A + B + C = 180^\circ$,

$$\begin{aligned}
 1 - \cos^2 B + \cos^2 A - \cos^2 C &= \sin^2 B + \sin^2 C - \sin^2 A \\
 &= \sin^2 B + \sin (C + A) \sin (C - A) \\
 &= \sin B \{ \sin (C + A) + \sin (C - A) \}, \\
 &\qquad\qquad\qquad \text{since } C + A = 180^\circ - B, \\
 &= 2 \sin B \sin C \cos A.
 \end{aligned}$$

When $A + B + C = 0$,

$$\begin{aligned}
 1 - \cos^2 B + \cos^2 A - \cos^2 C &= \sin^2 B + \sin (C + A) \sin (C - A) \\
 &= -\sin B \{ \sin (C + A) + \sin (C - A) \}, \\
 &\qquad\qquad\qquad \text{since } C + A = -B, \\
 &= -2 \sin B \sin C \cos A.
 \end{aligned}$$

268. We have $\cot A - \cot B = \cot B - \cot C$,

that is $\frac{\sin (A - B)}{\sin A \sin B} = \frac{\sin (B - C)}{\sin B \sin C}$,

or $\frac{\sin (A - B)}{\sin (B + C)} = \frac{\sin (B - C)}{\sin (A + B)}$.

Whence $\sin (A + B) \sin (A - B) = \sin (B + C) \sin (B - C)$,

$$\sin^2 A - \sin^2 B = \sin^2 B - \sin^2 C;$$

that is, $a^2 - b^2 = b^2 - c^2$.

269. Second side $= 2 \cos \frac{5a - 2\beta - \gamma}{4} \left\{ \cos \frac{4\beta + 3\gamma - 3a}{4} + \cos \frac{6\beta - 7\gamma + a}{4} \right\}$

$$\begin{aligned}
 &= \cos \frac{2a + 2\beta + 2\gamma}{4} + \cos \frac{6\beta + 4\gamma - 8a}{4} + \cos \frac{6a + 4\beta - 8\gamma}{4} \\
 &\qquad\qquad\qquad + \cos \frac{6\gamma + 4a - 8\beta}{4} \\
 &= \cos \frac{\pi}{2} + \cos \left(\frac{3\beta}{2} + \gamma - 2a \right) + \cos \left(\frac{3a}{2} + \beta - 2\gamma \right) \\
 &\qquad\qquad\qquad + \cos \left(\frac{3\gamma}{2} + a - 2\beta \right) \\
 &= \text{first side.}
 \end{aligned}$$

270. Denote the radii of the three escribed circles by x, y, z respectively, then we have to shew that

$$\begin{aligned}
 (y - z) (z - x) (x - y) + (y - z) (z + x) (x + y) + (z - x) (x + y) (y + z) \\
 + (x - y) (y + z) (z + x) = 0.
 \end{aligned}$$

Taking the terms in pairs, the expression on the left reduces to

$$(y - z) \{ 2 (zx + xy) \} + (y + z) \{ 2 (zx - xy) \},$$

or $2x (y - z) (y + z) + 2x (y + z) (z - y),$

which is identically equal to zero.

271. We have $32 \sin \frac{A}{2} \sin \frac{5A}{2} = 16 (\cos 2A - \cos 3A)$.

Now $\cos 2A = 2 \cos^2 A - 1 = \frac{2 \times 9}{16} - 1 = \frac{1}{8},$

$$\cos 3A = 4 \cos^3 A - 3 \cos A = \frac{4 \times 27}{64} - \frac{3 \times 3}{4} = -\frac{9}{16};$$

$$\therefore 32 \sin \frac{A}{2} \sin \frac{5A}{2} = 16 \left(\frac{1}{8} + \frac{9}{16} \right) = 11.$$

272. Solving the quadratic, we have $\tan \theta = -1 \pm \sqrt{2}$.

Now $\sqrt{2} - 1 = \tan \frac{\pi}{8}. \quad [\text{Art. 251.}]$

$$-(\sqrt{2} + 1) = -\cot \frac{\pi}{8} = -\tan \left(\frac{\pi}{2} - \frac{\pi}{8} \right).$$

From the first result, we get $\theta = n\pi + \frac{\pi}{8},$

and from the second, $\theta = n\pi - \left(\frac{\pi}{2} - \frac{\pi}{8} \right) = n\pi - \frac{3\pi}{8},$

both of which are included in $(8n-1) \frac{\pi}{8} \pm \frac{\pi}{4}.$

273. (1) $2 \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{\frac{2}{7}}{1 - \frac{1}{49}} = \tan^{-1} \frac{14}{48} = \tan^{-1} \frac{7}{24} = \cos^{-1} \frac{24}{25},$

$$4 \tan^{-1} \frac{1}{3} = 2 \tan^{-1} \frac{\frac{2}{3}}{1 - \frac{1}{9}} = 2 \tan^{-1} \frac{3}{4} = \tan^{-1} \frac{\frac{3}{2}}{1 - \frac{9}{16}} = \tan^{-1} \frac{24}{7}$$

$$= \sin^{-1} \frac{24}{25}.$$

Thus each side of the identity $= \frac{24}{25}.$

(2) $\frac{3 \sin 2a}{5 + 3 \cos 2a} = \frac{6 \tan a}{1 + \tan^2 a} \div \left\{ 5 + \frac{3(1 - \tan^2 a)}{1 + \tan^2 a} \right\} = \frac{3 \tan a}{4 + \tan^2 a};$

$$\begin{aligned} \therefore \text{first side of the identity} &= \tan^{-1} \frac{3 \tan a}{4 + \tan^2 a} + \tan^{-1} \left(\frac{\tan a}{4} \right) \\ &= \tan^{-1} \frac{\frac{3 \tan a}{4 + \tan^2 a} + \frac{\tan a}{4}}{1 - \frac{3 \tan^2 a}{4(4 + \tan^2 a)}} = \tan^{-1} \frac{16 \tan a + \tan^3 a}{16 + \tan^2 a} \\ &= \tan^{-1} (\tan a) = a. \end{aligned}$$

$$274. \text{ We have } \frac{s(s-a)}{bc} = \frac{b^2+c^2}{4bc};$$

$$\therefore 2s(2s-2a) = b^2+c^2;$$

$$\text{or } (b+c+a)(b+c-a) = b^2+c^2,$$

$$\text{that is, } b^2+c^2+2bc-a^2 = b^2+c^2;$$

$$\therefore \frac{a^2}{2} = bc,$$

which proves the proposition.

$$275. \text{ We have } \sin B = \frac{b}{a} \sin A = \frac{119 \sin 50^\circ}{97},$$

$$\log 119 = 2.0755470$$

$$\log \sin 50^\circ = \bar{1}.8842540$$

$$\hline 1.9598010$$

$$\log 97 = 1.9867717$$

$$\log \sin B = \bar{1}.9730293$$

$$\log \sin 70^\circ = \bar{1}.9729858$$

$$\hline 435$$

$$\text{diff. for } 1' = 460,$$

$$\frac{435}{460} \times 60'' = 57'';$$

$$\therefore B = 70^\circ 0' 57'', \text{ or } 109^\circ 59' 3'', \text{ both values being admissible since } a < b;$$

$$\therefore C = 59^\circ 59' 3'' \text{ or } 20^\circ 0' 57''.$$

$$276. \text{ The } \angle BD_1F_1 = \frac{1}{2} (\text{supp}^t \text{ of } \angle F_1BD_1) = \frac{B}{2} = \angle BF_1D_1.$$

$$\text{Similarly } \angle CD_1E_1 = \frac{C}{2};$$

$$\therefore \angle F_1D_1E_1 = 180^\circ - \frac{B+C}{2} = 90^\circ + \frac{A}{2}.$$

Again from the isosceles triangle AE_1F_1 ,

$$\angle AF_1E_1 = 90^\circ - \frac{A}{2}; \therefore \angle D_1F_1E_1 = 90^\circ - \frac{A+B}{2} = \frac{C}{2},$$

and similarly

$$\angle D_1E_1F_1 = \frac{B}{2}.$$

$$\text{Now } r_a = \frac{E_1F_1 \sin \frac{1}{2} E_1 \sin \frac{1}{2} F_1}{\cos \frac{1}{2} D_1} = \frac{2s \sin \frac{A}{2} \sin \frac{B}{4} \sin \frac{C}{4}}{\cos \left(45^\circ + \frac{A}{4} \right)}$$

$$= 4s \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{C}{4} \cdot \frac{\cos \frac{A}{4}}{\frac{1}{\sqrt{2}} \left(\cos \frac{A}{4} - \sin \frac{A}{4} \right)}$$

$$= 4\sqrt{2}s \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{C}{4} \cdot \frac{1}{1 - \tan \frac{A}{4}};$$

$$\begin{aligned}\therefore \frac{1}{r_a} : 1 - \tan \frac{A}{4} &= 1 : 4\sqrt{2} s \sin \frac{A}{4} \sin \frac{B}{4} \sin \frac{C}{4} \\ &= \frac{1}{r_b} : 1 - \tan \frac{B}{4} = \frac{1}{r_c} : 1 - \tan \frac{C}{4}\end{aligned}$$

by symmetry.

277. The expression $= \tan^{-1} \frac{x \cos \theta}{1 - x \sin \theta} - \tan^{-1} \frac{x - \sin \theta}{\cos \theta}$

and this reduces to $\tan^{-1} \left\{ \frac{\sin \theta (1 - 2x \sin \theta + x^2)}{\cos \theta (1 - 2x \sin \theta + x^2)} \right\}$,

which equals $\tan^{-1} (\tan \theta)$, or θ .

278. By Example XIII. *c.* 8 we have

$$\frac{\cos A + \cos B}{4 \sin^2 \frac{C}{2}} = \frac{a+b}{2c};$$

$\therefore a+b=2c$; whence a, c, b are in A.P.

279. The expression

$$\begin{aligned}&= 2 \cos a \cos \beta \{ \cos (\gamma + \delta) + \cos (\gamma - \delta) \} + 2 \sin a \sin \beta \{ \cos (\gamma - \delta) - \cos (\gamma + \delta) \} \\ &= 2 \cos (\gamma + \delta) \cos (a + \beta) + 2 \cos (\gamma - \delta) \cos (a - \beta) \\ &= \cos (a + \beta + \gamma + \delta) + \cos (a + \beta - \gamma - \delta) + \cos (a - \beta + \gamma - \delta) + \cos (a - \beta - \gamma + \delta).\end{aligned}$$

280. The $\angle BIC = 180^\circ - \frac{B+C}{2} = 90^\circ + \frac{A}{2}$;

$$\therefore \rho_1 = \frac{a}{2 \sin BIC} = \frac{a}{2 \cos \frac{A}{2}};$$

$$\begin{aligned}\therefore \rho_1 \rho_2 \rho_3 &= \frac{abc}{8 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{R^3 \sin A \sin B \sin C}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} \\ &= 8R^3 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 2rR^2,\end{aligned}$$

since

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

281. This is a particular case of Ex. 13 in XVII. *a.*

282. The equation may be written

$$b^2 \sin^2 2\theta = (c - a \cos 2\theta)^2,$$

or

$$b^2 (1 - \cos^2 2\theta) = c^2 - 2ac \cos 2\theta + a^2 \cos^2 2\theta,$$

that is,

$$(a^2 + b^2) \cos^2 2\theta - 2ac \cos 2\theta + c^2 - b^2 = 0;$$

therefore, by the theory of quadratic equations,

$$\cos 2\alpha + \cos 2\beta = \frac{2ac}{a^2 + b^2};$$

$$\therefore 2 \cos^2 \alpha - 1 + 2 \cos^2 \beta - 1 = \frac{2ac}{a^2 + b^2};$$

whence

$$\cos^2 \alpha + \cos^2 \beta = \frac{a^2 + ac + b^2}{a^2 + b^2}.$$

283. We have $c^2 = a^2 + b^2 - 2ab \cos C$

$$= 2 + 2 + \sqrt{2} - 2\sqrt{2} \sqrt{2 + \sqrt{2}} \cdot \frac{\sqrt{2 + \sqrt{2}}}{2}$$

$$= 4 + \sqrt{2} - \sqrt{2} (2 + \sqrt{2}) = 2 - \sqrt{2};$$

$$\therefore c = \sqrt{2 - \sqrt{2}}.$$

Now

$$\sin A = \frac{a}{c} \sin C = \frac{\sqrt{2}}{\sqrt{2 - \sqrt{2}}}, \quad \frac{\sqrt{2 - \sqrt{2}}}{2} = \frac{1}{\sqrt{2}};$$

therefore $A = 45^\circ$, or 135° , and since a is not the greatest side the smaller value must be taken.

Therefore

$$B = 112\frac{1}{2}^\circ.$$

284. We have

$$\sin 3A = 3 \sin A - 4 \sin^3 A;$$

$$\therefore \sin^3 A = \frac{3}{4} \sin A - \frac{1}{4} \sin^3 A;$$

$$\therefore \Sigma \sin^3 A = \frac{3}{4} \Sigma \sin A - \frac{1}{4} \Sigma \sin 3A.$$

$$\text{Now } \Sigma \sin A = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2},$$

$$\begin{aligned} \text{and } \Sigma \sin 3A &= 2 \sin \frac{3(A+B)}{2} \cos \frac{3(A-B)}{2} + 2 \sin \frac{3C}{2} \cos \frac{3C}{2} \\ &= 2 \sin \left(270^\circ - \frac{3C}{2} \right) \cos \frac{3(A-B)}{2} + 2 \sin \frac{3C}{2} \cos \frac{3C}{2} \\ &= -2 \cos \frac{3C}{2} \cos \frac{3(A-B)}{2} + 2 \sin \frac{3C}{2} \cos \frac{3C}{2} \end{aligned}$$

$$\begin{aligned}
&= 2 \cos \frac{3C}{3} \left\{ \sin \frac{3C}{2} - \cos \frac{3(A-B)}{2} \right\} \\
&= 2 \cos \frac{3C}{2} \left\{ -\cos \frac{3(A+B)}{2} - \cos \frac{3(A-B)}{2} \right\} \\
&= -4 \cos \frac{3A}{2} \cos \frac{3B}{2} \cos \frac{3C}{2}; \\
\therefore \Sigma \sin^3 A &= 3 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} - \cos \frac{3A}{2} \cos \frac{3B}{2} \cos \frac{3C}{2}.
\end{aligned}$$

285. Take the third figure on p. 131, and first suppose that
 $\angle AB_2C = 2 \angle AB_1C$.

Then it easily follows that $\triangle CB_1B_2$ is equilateral;

$$\therefore b \sin A = a \sin B \text{ becomes } b \sin A = \frac{\sqrt{3}}{2} a.$$

Secondly, suppose that $\angle ACB_1 = 2 \angle ACB_2$.

Then $\angle ACB_2 = \angle B_1 - \angle A = \angle B_2CB_1 = 180^\circ - 2 \angle B_1$;

$$\therefore 3B_1 = 180^\circ + A,$$

$$\sin 3B_1 + \sin A = 0; \text{ or } 3 \sin B_1 - 4 \sin^3 B_1 + \sin A = 0.$$

Substituting $\frac{b}{a} \sin A$ for $\sin B_1$, and reducing we obtain the required result.

286. If we write $\sqrt{y}$ in the place of x the resulting equation has $\tan^2 \alpha$, $\tan^2 \beta$, $\tan^2 \gamma$ for its roots. If in the last equation we further write z for $1+y$ the resulting equation has $\sec^2 \alpha$, $\sec^2 \beta$, $\sec^2 \gamma$ for its roots.

After making the above substitutions the equation in z is

$$z^3 - (p^2 + 3)z^2 + (4p^2 - 2pr + 3)z - (p-r)^2 - 1 = 0;$$

$$\therefore \sec^2 \alpha \sec^2 \beta \sec^2 \gamma = \text{product of the roots}$$

$$= (p-r)^2 + 1.$$

Otherwise. Let t_1, t_2, t_3 be the roots of the given equation; then

$$\begin{aligned}
\sec^2 \alpha \sec^2 \beta \sec^2 \gamma &= (1+t_1^2)(1+t_2^2)(1+t_3^2) \\
&= 1 + \Sigma t_1^2 + \Sigma t_1^2 t_2^2 + t_1^2 t_2^2 t_3^2,
\end{aligned}$$

$$\text{where } \Sigma t_1 = p, \Sigma t_1 t_2 = 0, t_1 t_2 t_3 = r.$$

$$\text{Thus } \sec^2 \alpha \sec^2 \beta \sec^2 \gamma = 1 + p^2 - 2pr + r^2.$$

287. Square the given equation, and write it in the form

$$\begin{aligned}
&\sin^2 \left(\frac{\pi}{4} - \frac{\alpha}{2} \right) \sin^2 \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \sin^2 \left(\frac{\pi}{4} - \frac{\gamma}{2} \right) \\
&= \cos^2 \left(\frac{\pi}{4} - \frac{\alpha}{2} \right) \cos^2 \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \cos^2 \left(\frac{\pi}{4} - \frac{\gamma}{2} \right),
\end{aligned}$$

$$\text{or} \quad (1 - \sin \alpha) (1 - \sin \beta) (1 - \sin \gamma) = (1 + \sin \alpha) (1 + \sin \beta) (1 + \sin \gamma);$$

$$\therefore \sin \alpha + \sin \beta + \sin \gamma + \sin \alpha \sin \beta \sin \gamma = 0,$$

$$\text{or} \quad 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} + 8 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\beta}{2} \sin \frac{\gamma}{2} \cos \frac{\gamma}{2} = 0;$$

$$\text{hence} \quad 1 + 2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = 0,$$

$$\text{that is,} \quad 1 + \frac{1}{2} (\cos \alpha + \cos \beta + \cos \gamma - 1) = 0;$$

$$\therefore \cos \alpha + \cos \beta + \cos \gamma + 1 = 0.$$

288. Each side of the heptagon subtends an angle $\frac{\pi}{7}$ at the circumference of the circle whose diameter is 2. Therefore if x represent a side,

$$x = 2 \sin \frac{\pi}{7}.$$

Now by Art. 331, the roots of the equation

$$8y^3 + 4y^2 - 4y - 1 = 0,$$

$$\text{are} \quad \cos \frac{2\pi}{7}, \quad \cos \frac{4\pi}{7}, \quad \cos \frac{6\pi}{7}.$$

Therefore $2 \cos \frac{2\pi}{7}$ satisfies $y^3 + y^2 - 2y - 1 = 0$.

Put $y = 2 - 4 \sin^2 \frac{\pi}{7} = 2 - x^2$ in this equation.

We obtain, after reduction,

$$x^6 - 7x^4 + 14x^2 - 7 = 0,$$

the roots of which are $2 \sin \frac{\pi}{7}$, $2 \sin \frac{2\pi}{7}$, $2 \sin \frac{3\pi}{7}$.

The first of these values corresponds to a side of the heptagon, the second and third to chords subtending at the circumference angles of $\frac{2\pi}{7}$ and $\frac{3\pi}{7}$ respectively. That is they represent the diagonals of the heptagon, as is easily seen from a figure.

289. We have $\cot(A + C) = -\cot B = -1$;

$$\therefore \frac{\cot A \cot C - 1}{\cot A + \cot C} = -1;$$

$$\text{that is,} \quad 1 + \cot A + \cot C + \cot A \cot C = 2,$$

$$\text{or} \quad (1 + \cot A) (1 + \cot B) = 2.$$

290. We have $8R^2 = a^2 + b^2 + c^2$
 $= (2R \sin A)^2 + (2R \sin B)^2 + (2R \sin C)^2$;
 $\therefore \sin^2 A + \sin^2 B + \sin^2 C = 2$;
 $\therefore 1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C = 4$,
 $\cos 2A + \cos 2B + \cos 2C + 1 = 0$;
 $\therefore 4 \cos A \cos B \cos C = 0$; [Examples XII. d. 9.]

therefore one of the angles must be a right angle.

291. We have $\sin B = \cos A \cos C$
 $= \cos A \tan C$,
 and $\cos B = \tan C$;
 $\therefore 1 = \tan^2 C (1 + \cos^2 A)$;
 $\cos^2 C = (1 - \cos^2 C) (1 + \cos^2 A)$,
 $\frac{\sin^2 A}{\cos^2 A} = \left(1 - \frac{\sin^2 A}{\cos^2 A}\right) (2 - \sin^2 A)$;
 or $\sin^2 A = (1 - 2 \sin^2 A) (2 - \sin^2 A)$;
 whence $\sin^4 A - 3 \sin^2 A + 1 = 0$;

$$\therefore \sin^2 A = \frac{3 - \sqrt{5}}{2},$$

the other value being impossible;

$$\therefore \sin^2 A = \frac{1}{4} (6 - 2\sqrt{5}),$$

or $\sin A = \frac{\sqrt{5} - 1}{2} = 2 \sin 18^\circ$
 $= \sin B = \sin C$ similarly.

292. It will be sufficient to prove that

$$ab + bc + ca = r^2 + s^2 + 4Rr \dots\dots\dots(1),$$

and $abc = 4Rrs \dots\dots\dots(2).$

Now $r^2 + s^2 + 4Rr = r(r_1 + r_2 + r_3) + s^2$ [XVIII. a. Ex. 25]

$$= \left(\frac{\Delta^2}{s(s-a)} + \dots + \dots \right) + s^2$$

$$= \{(s-b)(s-c) + \dots + \dots\} + s^2$$

$$= 4s^2 - 2s(a+b+c) + bc + ca + ab$$

$$= bc + ca + ab.$$

Again, $4Rrs = \frac{abc}{\Delta} \cdot \Delta = abc.$

$$293. \text{ We have } \sin \frac{\pi}{14} = \cos \left(\frac{\pi}{2} - \frac{\pi}{14} \right) = \cos \frac{3\pi}{7},$$

which is a root of the equation

$$8x^3 - 4x^2 - 4x + 1 = 0.$$

See solution of XXV. c. Ex. 16.

294. The distances of the successive heaps from the starting point are

$$2r \sin \frac{\pi}{n}, \quad 2r \sin \frac{2\pi}{n}, \quad 2r \sin \frac{3\pi}{n}, \quad \dots, \quad 2r \sin \frac{(n-1)\pi}{n};$$

$\therefore$ the whole distance traversed is twice the sum of this series.

$$\begin{aligned} \text{Now the sum of the sines} &= \frac{\sin \frac{(n-1)\pi}{2n} \sin \frac{1}{2} \left\{ \frac{\pi}{n} + \frac{(n-1)\pi}{n} \right\}}{\sin \frac{\pi}{2n}} \quad [\text{Art. 296.}] \\ &= \frac{\sin \left(\frac{\pi}{2} - \frac{\pi}{2n} \right) \sin \frac{\pi}{2}}{\sin \frac{\pi}{2n}} = \cot \frac{\pi}{2n}, \end{aligned}$$

whence the required result follows.

$$\begin{aligned} 295. \quad \cos \frac{\pi}{15} \cos \frac{4\pi}{15} &= \frac{1}{2} \left(\cos \frac{\pi}{3} + \cos \frac{\pi}{5} \right) = \frac{1}{2} \left(\frac{1}{2} + \frac{\sqrt{5}+1}{4} \right) = \frac{3+\sqrt{5}}{8}; \\ \cos \frac{2\pi}{15} \cos \frac{7\pi}{15} &= \frac{1}{2} \left(\cos \frac{\pi}{3} + \cos \frac{3\pi}{5} \right) = \frac{1}{2} \left(\frac{1}{2} - \frac{\sqrt{5}-1}{4} \right) = \frac{3-\sqrt{5}}{8}; \\ \cos \frac{3\pi}{15} \cos \frac{6\pi}{15} &= \frac{1}{2} \left(\cos \frac{\pi}{5} + \cos \frac{3\pi}{5} \right) = \frac{1}{2} \left(\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4} \right) = \frac{1}{4}; \\ \cos \frac{5\pi}{15} &= \cos \frac{\pi}{3} = \frac{1}{2}; \end{aligned}$$

$\therefore$ multiplying these results together, we have

$$\frac{1}{8} \left(\frac{3+\sqrt{5}}{8} \right) \left(\frac{3-\sqrt{5}}{8} \right) = \frac{4}{8^3} = \frac{2^2}{2^9} = \left(\frac{1}{2} \right)^7.$$

296. We have $ax + by + cz = 2\Delta$.

$$\begin{aligned} \text{Now } (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2 \\ = (bx - ay)^2 + (cy - bz)^2 + (az - cx)^2; \end{aligned}$$

$$\text{or } (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - 4\Delta^2 = (bx - ay)^2 + (cy - bz)^2 + (az - cx)^2.$$

$\therefore x^2 + y^2 + z^2$ is a minimum when the expression on the right is zero; that is when

$$bx = ay, \quad cy = bz, \quad az = cx;$$

$$\therefore \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{ax + by + cz}{a^2 + b^2 + c^2} = \frac{2\Delta}{a^2 + b^2 + c^2}.$$

297. With the notation and fig. of Art. 231, suppose AD and BC intersect in E , then r_a is the radius of the escribed circle opposite to E in the $\triangle ABE$;

$$\therefore r_a = \frac{a \cos \frac{A}{2} \cos \frac{B}{2}}{\sin \frac{A+B}{2}},$$

$$\therefore \frac{a}{r_a} = \tan \frac{A}{2} + \tan \frac{B}{2};$$

$$\begin{aligned} \therefore \frac{a}{r_a} + \frac{c}{r_c} &= \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} + \tan \frac{D}{2} \\ &= \frac{b}{r_b} + \frac{d}{r_d}, \text{ similarly.} \end{aligned}$$

298. Draw AH, AH' perpendicular to BC ; then

$$\angle PAH = (90^\circ - B) - (90^\circ - C) = C - B,$$

and

$$AH = 2R \sin B \sin C,$$

$$\therefore AP = \frac{2R \sin B \sin C}{\cos (C - B)};$$

$$\begin{aligned} \therefore \frac{1}{AP} + \dots + \dots &= \frac{1}{2R} \left\{ \frac{\cos (B - C)}{\sin B \sin C} + \dots + \dots \right\} \\ &= \frac{1}{4R} \cdot \frac{(\sin 2C + \sin 2B) + \dots + \dots}{\sin A \sin B \sin C} \\ &= \frac{2}{R}. \quad [\text{Art. 135, Ex. 1.}] \end{aligned}$$

Again, $BA' = 2R \cos BA'A = 2R \cos C$, since B, A', A, C are concyclic,

$$\therefore A'H' = 2R \cos B \cos C;$$

$$\begin{aligned} \therefore \frac{1}{A'P} + \dots + \dots &= \frac{1}{2R} \left\{ \frac{\cos (C - A)}{\cos B \cos C} + \dots + \dots \right\} \\ &= -\frac{1}{2R} \frac{(\cos 2C + \cos 2B) + \dots + \dots}{\cos A \cos B \cos C} \\ &= -\frac{2(\cos 2A + \cos 2B + \cos 2C)}{4R \cos A \cos B \cos C} \\ &= \frac{1}{2R} \cdot \frac{4 \cos A \cos B \cos C + 1}{\cos A \cos B \cos C}. \quad [\text{XII. d. Ex. 9.}] \end{aligned}$$

299. Let $\tan \frac{\theta}{2} = t$, then $\sin \theta = \frac{2t}{1+t^2}$, $\cos \theta = \frac{1-t^2}{1+t^2}$; substitute these values in the given equation; then after reduction, we obtain

$$t^4b + 2t^3(c-a) - 2t(c+a) - b = 0.$$

This equation has *four* roots; three of which are

$$\tan \frac{\alpha}{2}, \tan \frac{\beta}{2}, \tan \frac{\gamma}{2};$$

also

$$t_1 t_2 t_3 t_4 = -1, \quad t_2 t_3 + t_3 t_4 + \dots = 0.$$

Eliminating t_4 by means of these equations,

$$t_2 t_3 + t_3 t_1 + t_1 t_2 = \frac{1}{t_2 t_3} + \frac{1}{t_3 t_1} + \frac{1}{t_1 t_2};$$

$$\begin{aligned} \therefore \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} + \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \\ = \cot \frac{\beta}{2} \cot \frac{\gamma}{2} + \cot \frac{\gamma}{2} \cot \frac{\alpha}{2} + \cot \frac{\alpha}{2} \cot \frac{\beta}{2}; \end{aligned}$$

that is,

$$\Sigma \left(\tan \frac{\beta}{2} \tan \frac{\gamma}{2} - \cot \frac{\beta}{2} \cot \frac{\gamma}{2} \right) = 0;$$

$$\therefore \Sigma \frac{\sin^2 \frac{\beta}{2} \sin^2 \frac{\gamma}{2} - \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2}}{\sin \frac{\beta}{2} \sin \frac{\gamma}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}} = 0;$$

$$\therefore \Sigma \cos \frac{\beta+\gamma}{2} \cos \frac{\beta-\gamma}{2} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = 0;$$

$$\therefore \Sigma (\cos \beta + \cos \gamma) \sin \alpha = 0;$$

or

$$\Sigma \sin (\alpha + \beta) = 0.$$

300. By Example 2, Art. 331,

$$\sec^2 \frac{\pi}{7}, \quad \sec^2 \frac{2\pi}{7}, \quad \sec^2 \frac{3\pi}{7}$$

are roots of the equation

$$x^3 - 24x^2 + 80x - 64 = 0;$$

$$\therefore \sec^2 \frac{\pi}{7} + \sec^2 \frac{2\pi}{7} + \sec^2 \frac{3\pi}{7} = 24.$$

Also from XXV. c. Ex. 21,

$$\operatorname{cosec}^2 \frac{\pi}{7} + \operatorname{cosec}^2 \frac{2\pi}{7} + \operatorname{cosec}^2 \frac{3\pi}{7} = 8,$$

whence the required result follows.

